Linear Time-Invariant Systems, Behaviors and Modules

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Introduction

In Section 1.1 we explain the basic objects and goals of LTI (linear time-invariant) systems theory for the continuous-time standard case over the complex field in the behavioral language. In Section 1.9 we point out the other LTI-theories treated in this book, in particular the discrete-time theory over an arbitrary field, the module-behavior duality enabling the simultaneous treatment of all these cases. In Section 1.11 we mention various other systems theories that are not discussed in this book and are often more general and difficult and less developed.
All notions, models and theories of this book from systems theory and electrical engineering have been taken from the engineering literature, mainly from the textbooks [75], [39], [21], [19], [20], [73], [70], [72], [2], [66], [3], [13]. Further textbooks on the subjects of this book are [30], [68], [77], [35], [32], [36]. Many different mathematical subjects have been used in the original literature and in the cited books and are also applied, but often in a different form, in the present book. Among these fields are linear and polynomial algebra, module theory, differential equations, topology, convolution, distributions, Fourier series and transform, Laplace transform, cf. [39, XVII].
The goal of this book is

1. to derive the systems theoretic and electrical engineering results, mainly taken from or inspired by the quoted textbooks, by partly new mathematical methods, in particular by module-behavior duality.

2. to give complete and exact proofs of all results from item 1. and also of all used mathematical results that go beyond the first two years of university mathematics. The latter will be recalled, but without proofs.

3. to accompany all important results by algorithms that can be implemented in all computer algebra systems, for instance in MAPLE, and to demonstrate such implementations in several nontrivial examples that are mainly exposed in the later application Chapters 7, 9 and 11. The examples use and demonstrate many of the results and algorithms of the preceding chapters. The algorithms require, of course, an understanding of the meaning of the quantities that appear in them, but not of all mathematical details of their derivations. Application of the algorithms to various problems of the quoted textbooks will further demonstrate their applicability and usefulness.

Except that mentioned in 2. we do not assume any previous knowledge, in particular of systems or control theory, in contrast to several of the cited books. Like in most of these the exposition of the necessary mathematics requires substantial space in this book. This applies, in particular, to module-behavior

duality, quotient rings and modules and simplified versions of the last four analysis subjects from above. Of course, the study of this material can be omitted if the reader knows it.

The book can be studied by everyone, who is interested in the treated subjects, cf. the Contents and Chapter 1, and has the prerequisites mentioned in 2. In addition a certain so-called mathematical maturity, i.e, an experience with rigorous proofs and algorithms, is desirable. Knowledge of physics or engineering is not required, but of course an advantage. Electrical and translational mechanical networks are described from scratch, but other parts of mechatronics like electromechanical systems are not touched at all. For these we refer to the books on mechatronics, for instance [4], [38], [41]. From our study of the quoted engineering textbooks and from our experience as mathematicians we conclude that readers with either a mathematical or an engineering background will have no problems with the presupposed analysis, in particular with a higher order ordinary linear differential equation with constant coefficients and with elementary complex variables. The Laplace transform, in particular of the Dirac distribution and its derivatives, is assumed as standard knowledge of engineering students in most cited engineering textbooks, whereas it is not discussed at all in the regular curriculum of the first two years in mathematics. It is a difficult subject, cf. [67, Ch. VIII], [13, §12.3.4], and is therefore fully developed in this book by a new rigorous method that resembles Heaviside's unproven original operational calculus. In contrast, the algebraic prerequisites are standard knowledge of mathematicians, but, as far as we can infer from the quoted textbooks, not of engineers. So readers with this background will have to study some algebra that is recalled in the book, but without proofs, mainly basic definitions and results on (noetherian) commutative rings and modules, in particular Hom and exactness, principal ideal domains and the Smith form of polynomial matrices.

We refer to the quoted books for the history of systems theory, for much larger bibliographies than in the present book that also list the numerous original papers, for various introductions to the methods and results up to 1970 and the validity of the used models and their technical boundaries. Very many outstanding and well-known scientists contributed to the field, and their previous ideas and important work are, of course, also the basis of this book. Due to their large number we can only mention some of them. We refer to their homepages for bibliographical data. We do not quote the mathematical details of the original papers since ours are different in general. Expert systems theorists will, of course, recognize, how we adapted the ideas of our predecessors to our framework. In general we do not discuss this transformation process. For many results of the book we point, however, to corresponding results from the cited books, and the reader can thus compare the results and methods of these books with ours.

The *Contents* list the discussed subjects of the book. Chapter 1 is a detailed comment on the content and a self-contained survey over larger and, according to the cited textbooks and the engineering community, highly significant parts of linear time-invariant systems theory and electrical engineering and the decisive equations of these fields, on the basis of mathematical knowledge of two university years. In this chapter we present the most important methods and results of the book. We state the results and refer to the sections or theorems where they are discussed, and also to corresponding results in the cited books.

Of course, the chapter contains no proofs and does not assume any knowledge from the other chapters. Many advanced notions, methods and results have to be explained. We have done this in a mathematical language that is known after two years of studies in mathematics. All additional notions are introduced in the chapter. As the title *Survey* of Chapter 1 indicates its content will be discussed in detail in the later chapters and is, of course, not presupposed in these. So a potential reader need not read Chapter 1 to understand the following chapters. However, we recommend this.
Most of the book's results are constructive and accompanied by partly new algorithms, but the latter are not exposed in Chapter 1. They can be carried out with all computer algebra systems. The most important tools are the computations of the Smith form of polynomial matrices and of the complex roots and complex partial fraction decomposition of rational polynomials. Over the base fields of rational and Gaussian numbers as in all practical cases and over finite fields the Smith form is given precisely. Over the real or complex numbers problems with numerical computations may arise, but are not discussed in this book. As important applications we discuss *electrical and translational mechanical networks*. From an application point of view the following sections of this book are the most important ones:

4. The Sections 7.3 and 7.4 on *electrical and mechanical networks*, cf. their survey in Section 1.7. They furnish comprehensive tools for the analysis and synthesis of these networks, but we do not treat the vast field of synthesis of networks with specific properties. Theorems 7.3.11, 7.3.18, 7.3.21, 7.3.23, 7.3.32,7.3.40, 7.3.43 are the main results. Examples 7.3.17, 7.3.25, 7.3.33, 7.4.6, 7.4.7 demonstrate the algorithms and their implementation.

5. Section 9.2, cf. its survey in Section 1.8, on the construction and parametrization, for a stabilizable plant, of all *stabilizing feedback compensators that perform the tasks of tracking and disturbance rejection, and on their robustness. The main construction resp. robustness results are Theorems 9.2.8, 9.2.11, 9.2.17 resp. Theorems 9.2.32, 9.2.47, 9.2.50.* Examples 9.1.17, 9.1.22, 9.1.23 and 9.2.12 demonstrate the algorithms and their implementation.

6. In Section 11 we compute state space realizations of input/output behaviors by means of Gröbner bases, cf. Section 1.2. This method gives more general and more constructive results than in the literature. The Examples 11.3.11, 11.4.9, 11.4.11, 11.4.15 demonstrate the algorithms and their implementation.

7. In Chapter 12 we extend the standard *fractional calculus* considerably and solve complicated linear systems of generalized fractional integral/differential equations constructively. Theorem 12.1.3 is the main result and Example 12.1.7 gives simple, but instructive examples.

Compared to the existing literature and especially to the quoted textbooks essential and, in our opinion, of course, favorable modifications are carried out in the following subjects:

8. Behavior-module duality instead of time-frequency domain duality. In the latter the transformation from the time-domain to the frequency-domain is, in general, connected with a loss of information. This is avoided by the

categorical module-behavior duality. In particular, the fine properties of autonomous and noncontrollable behaviors can thus be studied.

9. Behavior isomorphisms instead of system equivalences. The categorical module-behavior duality enables this and simplifies the study of system equivalences.

10. Algebraic definition of the rational transfer matrix without the impulse response and without the Laplace transform. All standard properties of the transfer matrix hold and are proven.

11. Stability theory by means of the characteristic variety and of quotient modules.

12. (Periodic) distributions, Laplace transform and Fourier series and construction of transfer operators (input/output maps) without impulse responses and without integral operators. In its simplest and most important form the inverse Laplace transform describes the bijection of the predefined rational transfer matrices onto their (possibly distributional) impulse responses.

13. The input/output representation of an electrical or mechanical network by means of the simple Gauß algorithm instead of the usual tree-cotree graph theoretical methods and its study by means of the transfer matrix and operator.

14. The construction of stabilizing compensators and the study of their robustness by means of quotient signal modules.

15. State space realizations by means of Buchberger's Gröbner basis algorithm. This method gives more precise results than usual and is fully constructive since this algorithm is implemented in all computer algebra systems.

16. Generalized fractional calculus and behaviors via vector space-behavior duality and constructive solution of multivariable linear systems of generalized fractional integral/differential equations.

For the study of electrical networks in Section 7.3 most results of Chapters 2-7 are needed except those of Chapter 6, Section 7.1 and those on state space behaviors and on Rosenbrock equations in Sections 3.1.2-3.1.3, 5.3.2-5.3.5. In Chapter 9 the Chapters 6 resp. 8 on (feedback) interconnections resp. on stability via quotient modules are essential additional tools. For the application of the results to state space systems their previous study is, of course, required.
We use the standard notations $\mathbb{N}$ ($\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$) for the natural numbers (integers, rational numbers, real numbers, complex numbers). The real resp. imaginary part of a complex number z is denoted by $\Re(z)$ resp. $\Im(z)$. The number of elements of a finite set S is $\sharp(S)$. Other more specific notations are listed in the index of the book.

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Chapter 1

A survey of the book's content

1.1 Modules and behaviors

In Chapter 1 we explain the problems and results of LTI systems theory in the continuous-time case over the complex field $\mathbb{C}$. The theory for the real field $\mathbb{R}$, that is predominant in the engineering literature, is amply treated in the book. A complex polynomial in the indeterminate s has the form $f = \sum_{\mu=0}^{d} f_\mu s^\mu =: \sum_{\mu=0}^{\infty} f_\mu s^\mu,\ d \in \mathbb{N}$, where the f_μ belong to $\mathbb{C}$ and are zero for $\mu > d$. The letter s for the indeterminate comes from the Laplace transform where s denotes a complex number with sufficiently large real part. It also reminds of the shift operator in discrete-time systems theory. The set $\mathbb{C}[s]$ of all polynomials with the standard addition and multiplication is a principal ideal domain, and its quotient field $\mathbb{C}(s)$ consists of the rational functions $h(s) = f(s)g(s)^{-1},\ f, g \in \mathbb{C}[s],\ g \neq 0$.
Let $\mathcal{F}$ be a vector space of complex-valued functions $y(t), t \in \mathbb{R}$, on the real line $\mathbb{R}$. In systems theory and electrical engineering $\mathbb{R}$ resp. t are interpreted as the *time axis resp. a time instant*, and the function y is called a *signal*.

The basic equations for the considered theories are differential and require that $\mathcal{F}$ is closed under differentiation, i.e., $y \in \mathcal{F}$ implies $s \circ y := dy/dt \in \mathcal{F}$. The prototypical space with this property is the space $\mathrm{C}^\infty := \mathrm{C}^\infty(\mathbb{R}, \mathbb{C})$ of smooth complex-valued functions. This signal space is, however, too restricted for engineering applications, since these require piecewise continuous signals with jumps, for instance to describe the switching of electrical networks. The smallest space that contains these signals and is closed under differentiation is the space $\mathrm{C}^{-\infty} := \mathrm{C}^{-\infty}(\mathbb{R}, \mathbb{C})$ of *distributions of finite order* that consists of all derivatives of (piecewise) continuous signals, cf. Sections 1.3 and 7.2 for a detailed treatment. The derivative $d/dt : \mathrm{C}^{-\infty} \to \mathrm{C}^{-\infty}$ is defined as a $\mathbb{C}$-linear derivation such that dy/dt coincides with the standard derivative y' for continuously differentiable functions $y \left(\in \mathrm{C}^1\right)$. In particular, $\mathrm{C}^{-\infty}$ contains Dirac's δ-distribution

$$\delta := dY/dt = d^2y/dt^2,\ Y(t) := \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases},\ y(t) := \begin{cases} t & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}. \tag{1.1}$$

Here Y is *Heaviside's step function* and δ is interpreted as an impulse at $t = 0$, cf. (1.38). We define the *scalar multiplication*

$$f \circ y := \sum_{\mu=0}^{\infty} f_\mu y^{(\mu)},\ y^{(\mu)} := d^\mu y/dt^\mu,\ f = \sum_\mu f_\mu s^\mu \in \mathbb{C}[s],\ y \in \mathcal{F}, \tag{1.2}$$

that makes $\mathcal{F}$ a $\mathbb{C}[s]$-module, i.e., addition and scalar multiplication satisfy the associative, commutative and distributive laws like a vector space. So f acts on y as differential operator. The column space $\mathcal{F}^l$, $l \in \mathbb{N}$, is also a $\mathbb{C}[s]$-module with the componentwise structure. Consider a polynomial $k \times l$-matrix

$$R = (R_{\alpha\beta})_{1\le\alpha\le k,\ 1\le\beta\le l} \in \mathbb{C}[s]^{k\times l},\ k, l \in \mathbb{N},\ R_{\alpha\beta} = \sum_\mu R_{\alpha\beta,\mu} s^\mu \in \mathbb{C}[s]. \tag{1.3}$$

For a column vector $w = (w_1, \cdots, w_l)^\top \in \mathcal{F}^l$ we define

$$\begin{aligned} &R \circ w \in \mathcal{F}^k,\ (R \circ w)_\alpha := \sum_{\beta=1}^{l} R_{\alpha\beta} \circ w_\beta = \sum_{\beta=1}^{l} \sum_{\mu\in\mathbb{N}} R_{\alpha\beta,\mu} w_\beta^{(\mu)}, \text{ and then} \\ &\mathcal{B} := \left\{ w \in \mathcal{F}^l;\ R \circ w = 0 \right\} \\ &\quad = \left\{ w \in \mathcal{F}^l;\ \forall \alpha = 1, \cdots, k:\ \sum_{\beta=1}^{l} \sum_{\mu\in\mathbb{N}} R_{\alpha\beta,\mu} w_\beta^{(\mu)} = 0 \right\}. \end{aligned} \tag{1.4}$$

The equation $R \circ w = x$ with given right side $x \in \mathcal{F}^k$ represents an *inhomogeneous* (x arbitrary) resp. *homogeneous* ($x = 0$) *implicit system of linear differential equations with constant coefficients* $R_{\alpha\beta,\mu}$. The solution set $\mathcal{B}$ is a $\mathbb{C}[s]$-submodule of $\mathcal{F}^l$, i.e., closed under addition and scalar multiplication. Its elements are the *trajectories* of $\mathcal{B}$. According to Willems [74] these solution modules $\mathcal{B}$ are called *behaviors* in systems theory. In *Algebraic Analysis*, i.e., the algebraic theory of linear PDEs (partial differential equations), they were already extensively studied by Ehrenpreis, Malgrange, Palamodov [26], [49], [58], [8] in the beginning 1960s, both for distributional and for smooth signals. This theory was applied to multidimensional systems theory in [53]. The present book describes, in particular, the much simpler one-dimensional version of this theory. *One-dimensional* resp. *multidimensional* systems or behaviors are described by linear systems of *ordinary* resp. of *partial* differential or difference equations with constant coefficients.

Equation systems $R \circ w = x$ and their solution modules $\mathcal{B}$ occur naturally when large systems are composed of many components, that are described by basic and simple linear differential equations with constant coefficients. Such systems arise from physics and engineering, economics, biology etc., also from more general nonlinear systems by linearization. Our models have been taken from the cited books. Prototypical examples are *electrical and mechanical networks* that will be studied in detail in Sections 7.3 and 7.4. The primary interest of an engineer is the behavior $\mathcal{B}$ and its trajectories that can be measured, controlled etc. and show *how the system behaves*, hence the chosen terminology.

The matrix R gives rise to its row-submodule

$$\begin{aligned} &U := \mathbb{C}[s]^{1\times k} R := \sum_{\alpha=1}^{k} \mathbb{C}[s] R_{\alpha-} := \left\{ \sum_{\alpha=1}^{k} f_\alpha R_{\alpha-};\ f_\alpha \in \mathbb{C}[s] \right\} \subseteq \mathbb{C}[s]^{1\times l}, \\ &R_{\alpha-} := (R_{\alpha 1}, \cdots, R_{\alpha l}) \in \mathbb{C}[s]^{1\times l}, \end{aligned} \tag{1.5}$$

of the free module $\mathbb{C}[s]^{1\times l}$ of l-dimensional rows. The latter has the standard $\mathbb{C}[s]$-basis

$$\begin{aligned} &\delta_\beta := (0,\cdots,0,\overset{\beta}{1},0,\cdots,0),\ \beta=1,\cdots,l,\ \text{with} \\ &\xi=(\xi_1,\cdots,\xi_l)=\sum_{\beta=1}^{l}\xi_\beta\delta_\beta\in\mathbb{C}[s]^{1\times l}. \end{aligned} \tag{1.6}$$

The α-th row resp. β-th column of the matrix R are denoted by $R_{\alpha-}=R_{\alpha,-}$ resp. by $R_{-\beta}=R_{-,\beta}$. The submodule U, in turn, induces the finitely generated factor module

$$\begin{aligned} &M:=\mathbb{C}[s]^{1\times l}/U:=\{\overline{\xi}:=\xi+U;\ \xi\in\mathbb{C}[s]^{1\times l}\}=\sum_{\beta=1}^{l}\mathbb{C}[s]\overline{\delta_\beta}\ \text{with} \\ &\overline{\xi}+\overline{\eta}:=\overline{\xi+\eta},\ f\overline{\xi}:=\overline{f\xi},\ \xi,\eta\in\mathbb{C}[s]^{1\times l},\ f\in\mathbb{C}[s], \end{aligned} \tag{1.7}$$

and the distinguished list of generators $\overline{\delta_\beta}$ that satisfy the relations

$$0=\overline{R_{\alpha-}}=\sum_{\beta=1}^{l}R_{\alpha\beta}\overline{\delta_\beta},\ \alpha=1,\cdots,k. \tag{1.8}$$

It was a simple, but important observation of Malgrange in 1962 that the map

$$\mathrm{sol}_{\mathcal{F}}(M):=\mathrm{Hom}_{\mathbb{C}[s]}(M,\mathcal{F})\overset{\cong}{\to}\mathcal{B},\ \phi\mapsto w,\ w_\beta:=\phi(\overline{\delta_\beta}), \tag{1.9}$$

is well-defined and a $\mathbb{C}[s]$-isomorphism, where $\mathrm{Hom}_{\mathbb{C}[s]}(M_1,M_2)$ denotes the $\mathbb{C}[s]$-module of all $\mathbb{C}[s]$-linear maps from a $\mathbb{C}[s]$-module M_1 into another one M_2. The isomorphism (1.9) is the first link between modules and behaviors. Equation (1.9) also was an early explicit appearance of solution modules that were later called behaviors by Willems.

For many important signal modules $\mathcal{F}$ there is a one-one correspondence between $\mathcal{B}$, U and M. This makes the old and well-established theory of polynomial matrices and finitely generated polynomial modules available for systems theory. The same algebraic theory was used by Kalman in the 1960s to derive his *state space theory* [40], by Rosenbrock [63] and Wolovich [75] in the 1970s for the *polynomial matrix models* or *differential operator representations* and also by Willems in his *theory of behaviors* [74], [60]. Indeed, there is no approach to LTI systems theory without univariate polynomial and rational matrices. In the so-called *frequency domain* the latter appear as rational Laplace transforms. In this book the frequency domain is replaced by the *algebraic domain* of finitely generated polynomial modules. *All* important algorithms of LTI systems theory in engineering, in systems theory, in the quoted and in the present book rest on algorithms from univariate polynomial algebra or, in the case of state space theory, also from linear algebra over a field. This explains why *algebra* plays such a dominant part in LTI systems theory. However, *analysis* is also an essential ingredient of the theory, of the quoted books and also here. Sections 7.2 and 9.2.3- 9.2.5 introduce and discuss, with complete and exact proofs, indispensible notions like distributions, in particular periodic ones, Laplace transform, convolution, Fourier series and integral and normed linear spaces. Lebesgue's theory, i.e., measure, integral and convolution, is not needed or used in this book.

We isolate two properties of $\mathcal{F}$ that imply the one-one correspondence $M \leftrightarrow U \leftrightarrow \mathcal{B}$. Since the rows of R generate U it is obvious that

$$\mathcal{B} = \left\{w \in \mathcal{F}^l;\ R \circ w = 0\right\} = U^\perp := \left\{w \in \mathcal{F}^l;\ \forall \xi \in U:\ \xi \circ w = 0\right\}, \tag{1.10}$$

i.e., $\mathcal{B}$ depends on U and M, but not on the special generating matrix R. The number $p := \operatorname{rank}(R)$ is the rank of R as matrix with entries in the field $\mathbb{C}(s)$. Since $\mathbb{C}[s]$ is a principal ideal domain, the $\mathbb{C}[s]$-module U is free of dimension p, i.e., has a basis of this length. In other words, there is a matrix $\widetilde{R} \in \mathbb{C}[s]^{p\times l}$ of

$$\operatorname{rank}(\widetilde{R}) = \operatorname{rank}(R) = p \text{ such that } U = \mathbb{C}[s]^{1\times k}R = \mathbb{C}[s]^{1\times p}\widetilde{R} = \oplus_{\alpha=1}^{p}\mathbb{C}[s]\widetilde{R}_{\alpha-}.$$

In the sequel we may and do therefore assume that $R = \widetilde{R}$, i.e., that $p = k$ and that the p rows of R are linearly independent and thus a $\mathbb{C}[s]$-basis of U.
Since R has rank p, there are various choices of p linearly independent columns of R. After such a choice and a possible permutation of the columns of R and the components of w we may assume that R, w and $\mathcal{B}$ have the form

$$\begin{aligned} &R = (P, -Q) \in \mathbb{C}[s]^{p\times(p+m)},\ m := l - p,\ \operatorname{rank}(P) = p \text{ or } \det(P) \neq 0,\\ &w = \left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right) \in \mathcal{F}^{p+m},\ \mathcal{B} = \left\{\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right) \in \mathcal{F}^{p+m};\ P \circ y = Q \circ u\right\}\\ &\Longrightarrow H := P^{-1}Q \in \mathbb{C}(s)^{p\times m}. \end{aligned} \tag{1.11}$$

Such a decomposition of R and $\mathcal{B}$ is called an *IO (input/output) decomposition* or *structure* with $u \in \mathcal{F}^m$ as *input* and $y \in \mathcal{F}^p$ as *output*, and $\mathcal{B}$ with this structure is called an *IO behavior*. The number m is called the *rank* of M and of $\mathcal{B}$. In engineering the input u is also called the *external excitation* or *cause* and y the *response, reaction* or *effect*. This interpretation and language is appropriate only if the input u is *free*, i.e., if *each* $u \in \mathcal{F}^m$ gives rise to an output $y \in \mathcal{F}^p$, i.e., a solution of $P \circ y = Q \circ u$. A (signal) module $\mathcal{F}$ is called *injective* if this holds, i.e., if all equations $P \circ y = Q \circ u$ with given $u \in \mathcal{F}^m$, $(P, -Q) \in \mathbb{C}[s]^{p\times(p+m)}$ and $\operatorname{rank}(P) = p$ have a solution y. Since $\mathbb{C}[s]$ is a principal ideal domain, it suffices that this holds for $p = m = 1$. In particular, every $\mathbb{C}(s)$-vector space is an injective $\mathbb{C}[s]$-module. We will study injectivity in detail in Section 2.2. *In the rest of Chapter 1 we assume that* ${}_{\mathbb{C}[s]}\mathcal{F}$ *is injective.*
The following three important signal modules are injective, cf. Results 2.2.12, 7.2.28 and 4.3.10:

$$\begin{aligned} &\mathcal{F} := \mathrm{C}^{-\infty} = \mathrm{C}^{-\infty}(\mathbb{R}, \mathbb{C}) \supset \mathrm{C}^{\infty} = \mathrm{C}^{\infty}(\mathbb{R}, \mathbb{C}) \supset \mathrm{t}(\mathcal{F})\\ &\mathrm{t}(\mathcal{F}) = \mathrm{t}(\mathrm{C}^{-\infty}) = \mathrm{t}(\mathrm{C}^{\infty}) = \bigoplus_{\lambda\in\mathbb{C}} \mathbb{C}[t]e^{\lambda t} = \bigoplus_{\lambda\in\mathbb{C}}\bigoplus_{k\in\mathbb{N}} \mathbb{C}t^k e^{\lambda t}. \end{aligned} \tag{1.12}$$

The module $\mathrm{t}(\mathcal{F})$ is the *torsion submodule* of $\mathcal{F}$ of all signals y that satisfy a differential equation $f \circ y = 0,\ 0 \neq f \in \mathbb{C}[s]$. It consists of the *polynomial-exponential* functions that are (finite) $\mathbb{C}$-linear combinations of functions $t^k e^{\lambda t}$, cf. Section 4.3.3. The following inclusions hold:

$$\begin{aligned} \mathrm{C}^{-\infty} &\supset \mathrm{C}^{0,\mathrm{pc}} := \mathrm{C}^{0,\mathrm{pc}}(\mathbb{R}, \mathbb{C}) := \{u : \mathbb{R} \to \mathbb{C} \text{ piecewise continuous}\}\\ &\supset \mathrm{C}^{0} := \mathrm{C}^{0}(\mathbb{R}, \mathbb{C}) := \{u : \mathbb{R} \to \mathbb{C} \text{ continuous}\}. \end{aligned} \tag{1.13}$$

The $\mathbb{C}[s]$-submodule $\mathrm{C}_+^{-\infty}$ of distributions *with left bounded support* is given by the derivatives of continuous functions with such support, i.e.,

$$\begin{aligned} &\mathrm{C}_+^{-\infty} := \mathrm{C}^{-\infty}(\mathbb{R},\mathbb{C})_+ := \bigcup_{n\geq 0} s^n \circ \mathrm{C}_+^0 \text{ where} \\ &\mathrm{C}_+^{0,(\mathrm{pc})} := \mathrm{C}^{0,(\mathrm{pc})}(\mathbb{R},\mathbb{C})_+ := \left\{u \in \mathrm{C}^{0,(\mathrm{pc})};\ \exists t_0 \forall t \leq t_0 :\ u(t) = 0\right\}. \end{aligned} \tag{1.14}$$

For an obvious reason the signals in $\mathrm{C}_+^{-\infty}$ are called *initially-at-rest*. Since all technical systems start at some time t_0, mostly chosen as $t_0 = 0$, these signals are important. The δ-distribution according to (1.1) is obviously contained in $\mathrm{C}_+^{-\infty}$. The $\mathbb{C}[s]$-module $\mathrm{C}_+^{-\infty}$ is a $\mathbb{C}(s)$-vector space by a, necessarily unique, extension of the $\mathbb{C}[s]$-scalar multiplication, cf. Theorem 7.2.37. If u is continuous and zero for $t \leq t_0$ and $0 \neq f \in \mathbb{C}[s],\ d := \deg_s(f) :=$ degree of f, then $y := f^{-1} \circ u$ is the unique, d times continuously differentiable solution

$$y \in \mathrm{C}^d(\mathbb{R},\mathbb{C}) \text{ of } f \circ y = u \text{ with } y^{(\mu)}(t_0) = 0,\ \mu = 0, \cdots, d-1, \Longrightarrow y|_{(-\infty,t_0]} = 0.$$

A very important $\mathbb{C}(s)$-subspace of $\mathrm{C}_+^{-\infty}$ and thus $\mathbb{C}[s]$-injective is

$$\mathcal{F}_2 := \mathbb{C}[s] \circ \delta \oplus \mathrm{t}(\mathcal{F})Y,\ \mathbb{C}[s] \circ \delta = \oplus_{k\in\mathbb{N}} \mathbb{C}\delta^{(k)},\ \delta^{(k)} := s^k \circ \delta = d^k\delta/dt^k. \tag{1.15}$$

Signals $\alpha Y,\ \alpha \in \mathrm{t}(\mathcal{F})$, occur if a polynomial-exponential signal α is started at $t = 0$. So $\mathcal{F}_2$ consists of sums of such signals and $\mathbb{C}$-linear combinations of the derivatives of the Dirac distribution δ. *All these injective signal modules will be studied in Section 7.2.4.*

The next property of $\mathcal{F}$ ensures that $\mathcal{B} = U^\perp$ contains as much information as U. The behavior $\mathcal{B} = U^\perp \subseteq \mathcal{F}^l$ induces its orthogonal submodule

$$U^{\perp\perp} = \mathcal{B}^\perp := \left\{\xi \in \mathbb{C}[s]^{1\times l};\ \xi \circ \mathcal{B} = 0\right\} \supseteq U. \tag{1.16}$$

The trivial case $\mathcal{F} = 0$ and $U^{\perp\perp} = \mathbb{C}[s]^{1\times l}$ shows that $U^{\perp\perp} = U$ need not hold. The injective signal module $\mathcal{F}$ is called a *cogenerator*, cf. Section 2.3, if $U^{\perp\perp} = U$ holds for all submodules $U \subseteq \mathbb{C}[s]^{1\times l},\ l \in \mathbb{N}$, i.e., if U is determined by $\mathcal{B}$. This condition obviously implies and is indeed equivalent to the equivalences

$$\mathcal{B} = U^\perp = 0 \Longleftrightarrow U = \mathbb{C}[s]^{1\times l} \Longleftrightarrow M = 0. \tag{1.17}$$

The modules $\mathcal{F} = \mathrm{C}^{-\infty},\ \mathrm{C}^\infty,\ \mathrm{t}(\mathcal{F})$ are injective cogenerators. A $\mathbb{C}(s)$-vector space, for instance $\mathcal{F}_2$, is never a $\mathbb{C}[s]$-cogenerator. The direct sum module

$$\mathcal{F}_4 := \mathcal{F}_2 \bigoplus \mathrm{t}(\mathcal{F}) = \mathbb{C}[s] \circ \delta \bigoplus \left(\oplus_{\lambda\in\mathbb{C}} \mathbb{C}[t]e^{\lambda t}\right) Y \bigoplus \oplus_{\lambda\in\mathbb{C}} \mathbb{C}[t]e^{\lambda t}, \tag{1.18}$$

however, is injective and contains the cogenerator $\mathrm{t}(\mathcal{F})$, and is thus an injective cogenerator too. It consists of sums of signals in $\mathcal{F}_2$ and of polynomial-exponential signals. All signals in $\mathcal{F}_4$ are described by finitely many complex numbers and are especially suitable for computation. In *electrical engineering* signals in $\mathcal{F}_4$ and piecewise continuous periodic signals, see Section 1.4, are used almost exclusively. Less important signal modules $\mathcal{F}_1$ and $\mathcal{F}_3$ will be introduced in Section 7.2.4.

In the rest of this chapter we assume an injective cogenerator signal module $\mathcal{F}$ and describe further important consequences of this assumption. With

$R, U, M, \mathcal{B}$ from above the modules U resp. M are called the *equation resp. the system module* of $\mathcal{B}$.
For the behavior $\mathcal{B}$ from (1.4) and an arbitrary matrix $T \in \mathbb{C}[s]^{l_2 \times l}$ the injectivity of $\mathcal{F}$ implies that also the image $T \circ \mathcal{B}$ is a behavior, cf. Theorem 2.2.20. An important case of this is Willems' *elimination of latent variables* [60, Ch. 6]. Assume, more generally, two behaviors

$$\mathcal{B}_i = U_i^{\perp} \subseteq \mathcal{F}^{l_i},\ U_i \subseteq \mathbb{C}[s]^{1 \times l_i},\ M_i := \mathbb{C}[s]^{1 \times l_i}/U_i,\ i = 1, 2. \tag{1.19}$$

A $\mathbb{C}$-linear map $\phi :\ \mathcal{B}_1 \to \mathcal{B}_2$ is called a *behavior morphism* if there is a matrix $T \in \mathbb{C}[s]^{l_2 \times l_1}$ such that $\phi(w_1) = T \circ w_1$ for all $w_1 \in \mathcal{B}_1$, i.e., that ϕ is a differential operator. The set $\mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_2)$ of all these morphisms is a proper (cf. Example 2.3.17) $\mathbb{C}[s]$-submodule of $\mathrm{Hom}_{\mathbb{C}[s]}(\mathcal{B}_1, \mathcal{B}_2)$. The injective cogenerator property of $\mathcal{F}$ implies the canonical $\mathbb{C}[s]$-isomorphism, cf. Theorem 2.3.18,

$$\begin{aligned} &\mathrm{Hom}_{\mathbb{C}[s]}(M_2, M_1) \cong \mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_2),\ F \leftrightarrow \phi,\ T \in \mathbb{C}[s]^{l_2 \times l_1}, \\ &F(\xi_2 + U_2) = \xi_2 T + U_1,\ \phi(w_1) = T \circ w_1,\ \xi_2 \in \mathbb{C}[s]^{1 \times l_2},\ w_1 \in \mathcal{B}_1. \end{aligned} \tag{1.20}$$

This isomorphism implies that the bijective correspondence $M \leftrightarrow \mathcal{B}$ is very strong. It is called a *categorical duality* and is discussed in Section 2.3.3. In particular, ϕ is injective (surjective, bijective) if and only if F is surjective (injective, bijective). In this book behavioral isomorphisms $\phi : \mathcal{B}_1 \xrightarrow{\cong} \mathcal{B}_2$ and the dual isomorphisms F replace the various *system equivalences* in the literature, for instance Rosenbrock's and Fuhrmann's, cf. [39, pp. 561-566], [72, §2.2,§2.3]. If ϕ is injective, the implication $\phi(w_1) = \phi(\widetilde{w}_1) \Longrightarrow w_1 = \widetilde{w}_1$ suggested the language that w_1 *is observable from* $\phi(w_1)$. If there is a surjective $\phi : \mathcal{F}^{l_1} \to \mathcal{B}_2,\ \mathcal{B}_1 := 0^{\perp} = \mathcal{F}^{l_1}$, the behavior $\mathcal{B}_2$ is called *controllable* and ϕ is an *image representation* of $\mathcal{B}_2$. The term *controllable* is justified by Kalman's Theorem 3.3.10 and Willems' Theorem 3.3.4. If in this case $\phi(w_1) = w_2$ Pommaret calls w_1 a *potential* of w_2, a terminology suggested by an analogue for partial differential equations. The surjection ϕ implies the injection $F : M_2 \to M_1 = \mathbb{C}[s]^{1 \times l_1}/0 = \mathbb{C}[s]^{1 \times l_1}$ and thus that M_2 as submodule of a free $\mathbb{C}[s]$-module is itself free of dimension $m_2 := \mathrm{rank}(\mathcal{B}_2)$. Hence there is even a bijective image representation $\mathcal{F}^{m_2} \cong \mathcal{B}_2$. Thus a behavior $\mathcal{B}$ is controllable if and only if its module M is free. Observability and controllability are studied in Chapter 3. The main application of controllability in this book is for the construction and parametrization of *stabilizing compensators* in Chapter 9. Observability is a necessary and sufficient condition for the construction and parametrization of *functional observers* in Chapter 10.
LTI systems theory has three primary tasks and goals, cf. [21, §1-1]:

(i) *Modelling*: The theory of this book applies if a real world system can be described (approximately) by equations $R \circ w = x$ as in (1.4). Our models have been taken from the cited books.

(ii) *Analysis, both qualitative and quantitative*, i.e., to determine the properties of a given $\mathcal{B}$ by means of the properties of R, U and M and to compute numerical solutions.

(iii) *Synthesis or design*, i.e., to construct a behavior $\mathcal{B}$ with chosen properties, mainly of its transfer matrix H and its transfer operator, see Section 1.2.

In this book, synthesis is mainly treated in Chapter 9 where we discuss the construction of *stabilizing compensators* with special properties, mainly *tracking and disturbance rejection*. Kalman's realization theorem of proper transfer matrices, see (1.28) below, is also a synthesis result, cf. [21, §6.1]. The analysis, but not the synthesis of electrical and mechanical networks, for instance of filters, is treated in Section 7.3.

1.2 The transfer matrix and transfer operator

The assumptions of the preceding section are in force.
The IO behavior (1.11) implies the behavior isomorphism

$$\begin{aligned}&\mathcal{B}^0 := \{y \in \mathcal{F}^p;\ P \circ y = 0\} \cong \mathcal{B} \bigcap (\mathcal{F}^p \times \{0\}) = \{\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right) \in \mathcal{B};\ u = 0\},\ y \mapsto \left(\begin{smallmatrix} y \\ 0 \end{smallmatrix}\right),\\ &\Longrightarrow \mathrm{Hom}_{\mathbb{C}[s]}(M^0, \mathcal{F}) \cong \mathcal{B}^0 \text{ with } M^0 := \mathbb{C}[s]^{1\times p}/U^0,\ U^0 := \mathbb{C}[s]^{1\times p}P.\end{aligned} \tag{1.21}$$

A behavior (1.4) or (1.11) is called *autonomous* if it has no free components u or if the following equivalent properties hold, cf. Section 3.2:

$$\begin{aligned}&\mathrm{rank}(\mathcal{B}) = m = 0 \Longleftrightarrow \mathrm{rank}(R) = p = l \Longleftrightarrow P = R \Longleftrightarrow \mathcal{B}^0 = \mathcal{B}\\ &\Longleftrightarrow M = \mathrm{t}(M) \Longleftrightarrow \mathcal{B} = \mathrm{t}(\mathcal{B}) \Longleftrightarrow \dim_{\mathbb{C}}(M) < \infty \Longleftrightarrow \dim_{\mathbb{C}}(\mathcal{B}) < \infty\\ &\Longrightarrow \dim_{\mathbb{C}}(M) = \dim_{\mathbb{C}}(\mathcal{B}) \text{ and } \mathcal{B} \subset \mathrm{t}(\mathcal{F})^p = \oplus_{\lambda\in\mathbb{C}}\mathbb{C}[t]^p e^{\lambda t}\end{aligned} \tag{1.22}$$

where $\dim_{\mathbb{C}}(V)$ denotes the $\mathbb{C}$-dimension of a $\mathbb{C}$-vector space. Since $P \in \mathbb{C}[s]^{p\times p}$ and $\mathrm{rank}(P) = p$ the behavior $\mathcal{B}^0 = \{y \in \mathcal{F}^p;\ P \circ y = 0\}$ is autonomous and called the *autonomous or zero-input part* of $\mathcal{B}$. It depends on $\mathcal{B}$ and its IO structure, but not on the special choice of the defining matrices. Its dimension is

$$n := \dim_{\mathbb{C}}(\mathcal{B}^0) = \dim_{\mathbb{C}}(M^0) = \deg_s(\det(P)), \tag{1.23}$$

cf. Theorem 3.2.14. If y_1, y_2 are two outputs to the same input u, then $P \circ (y_2 - y_1) = Q \circ u - Q \circ u = 0$ implies $y_2 - y_1 \in \mathcal{B}^0$ or $y_2 = y_1 + z,\ z \in \mathcal{B}^0$.

The rational matrix $H := P^{-1}Q$ from (1.11) depends on U or $\mathcal{B}$ and the chosen IO decomposition, but again not on the special choice of the matrix $R = (P, -Q)$, cf. Theorem and Definition 5.2.2. It is called the *transfer matrix* of $\mathcal{B}$ and, for $p = m = 1$, the *transfer function* , and $\mathcal{B}$ is called an *IO realization* of H. A given rational matrix $H \in \mathbb{C}(s)^{p\times m}$ trivially admits various representations $H = P^{-1}Q$ with $(P, -Q) \in \mathbb{C}[s]^{p\times(p+m)}$ and $\mathrm{rank}(P) = p$, for instance $H = (f\,\mathrm{id}_p)^{-1}(fH)$ where f is a common denominator of the entries of H. Hence there are many IO behavior realizations of H, but only one *controllable realization*, cf. Corollary 5.2.3, that furnishes the, essentially unique, so-called *left coprime factorization* $H = P^{-1}Q$ of H.
Recall that $\mathrm{C}_+^{-\infty}$ is a $\mathbb{C}(s)$-vector space, and hence

$$H\circ : \left(\mathrm{C}_+^{-\infty}\right)^m \to \left(\mathrm{C}_+^{-\infty}\right)^p,\ u \mapsto H \circ u = \left(\sum_{\mu=1}^{m} H_{\nu\mu} \circ u_\mu\right)_{\nu=1,\cdots,p}, \tag{1.24}$$

is defined. This is the *transfer operator or IO map* induced by H. Note that the equation $P \circ y = Q \circ u$ does not define a linear map $u \mapsto y$ since, for general

$u \in (\mathrm{C}^{-\infty})^m$, y always exists, but is unique only up to a summand in $\mathcal{B}^0$. The map, cf. Theorem and Definition 7.2.41,

$$\left(\begin{smallmatrix} H \\ \mathrm{id}_m \end{smallmatrix}\right) \circ : \ \left(\mathrm{C}_+^{-\infty}\right)^m \cong \mathcal{B} \bigcap \left(\mathrm{C}_+^{-\infty}\right)^{p+m}, \ u \mapsto \left(\begin{smallmatrix} H \circ u \\ u \end{smallmatrix}\right), \tag{1.25}$$

is a $\mathbb{C}(s)$-isomorphism, and shows that the transfer matrix determines and is determined by the *initially-at-rest-part* of the IO behavior $\mathcal{B}$. If in this situation y is any other output to u, then

$$y_{ss} := H \circ u \text{ resp. } z := y - y_{ss} \in \mathcal{B}^0 \tag{1.26}$$

are often called the *steady or stationary state* resp. the *transient* of y. This language is appropriate only if $\mathcal{B}^0$ is *asymptotically stable*, i.e.,

$$\lim_{t \to \infty} z(t) = 0, \ z \in \mathcal{B}^0, \qquad \text{(cf. Section 1.6)} \tag{1.27}$$

so that y and $y_{ss} = H \circ u$ can be identified for large t, written as $y \approx y_{ss}$. Mainly in electrical engineering the linear equation

$$y \approx y_{ss} = H \circ u \text{ or } y_\nu \approx y_{ss,\nu} = \sum_{\mu=1}^{m} H_{\nu\mu} \circ u_\mu, \ \nu = 1, \cdots, p,$$

then establishes the *superposition principle*, experimentally due to Helmholtz: The partial effects $H_{\nu\mu} \circ u_\mu$ of the different input components u_μ are added (=superposed) to form the total effect of all input components on the output component $y_\nu \approx y_{ss,\nu}$. Notice that this principle does not apply to arbitrary equations $P \circ y = Q \circ u$. In the engineering literature [70], [66] the superposition principle is essentially used and experimentally or heuristically proved, but, in general, not with all necessary mathematical details.
Any IO behavior admits a *state space representation* as follows, cf. [40], Theorem and Definition 5.3.8 and Chapter 11: There are matrices

$$\begin{aligned} &A \in \mathbb{C}^{n \times n}, \ B \in \mathbb{C}^{n \times m}, \ C \in \mathbb{C}^{p \times n} \text{ and } D \in \mathbb{C}[s]^{p \times m} \text{ such that} \\ &\left(\begin{smallmatrix} C & D \\ 0 & \mathrm{id}_m \end{smallmatrix}\right) \circ : \ \mathcal{B}_s := \left\{ \left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in \mathcal{F}^{n+m}; \ s \circ x = Ax + Bu \right\} \\ &\quad = \left\{ \left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in \mathcal{F}^{n+m}; \ (s\,\mathrm{id}_n - A) \circ x = Bu \right\} \cong \mathcal{B}, \ \left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} Cx + D \circ u \\ u \end{smallmatrix}\right), \end{aligned} \tag{1.28}$$

is a behavior isomorphism. The matrices A, B, C are unique up to similarity, and D is unique. This means that if two quadrupels (A_i, B_i, C_i, D_i), $i = 1, 2$, of dimensions n_i, $i = 1, 2$, satisfy (1.28) then $n_1 = n_2 =: n$, and there is an invertible matrix

$$T \in \mathrm{Gl}_n(\mathbb{C}) \text{ such that } (A_2, B_2, C_2, D_2) = (TA_1T^{-1}, TB_1, C_1T^{-1}, D_1). \tag{1.29}$$

The behavior $\mathcal{B}_s$ is an IO behavior since the characteristic polynomial $\chi_A := \det(s\,\mathrm{id}_n - A)$ of A has degree n and is nonzero. The transfer matrices of $\mathcal{B}_s$ resp. $\mathcal{B}$, cf. Theorem and Definition 5.3.1, are

$$H_s = (s\,\mathrm{id}_n - A)^{-1}B \text{ resp. } H = P^{-1}Q = D + CH_s = D + C(s\,\mathrm{id}_n - A)^{-1}B. \tag{1.30}$$

If u is a piecewise continuous input, the vector x is continuous, and x and y have the standard form

$$\begin{aligned} x(t) &= e^{(t-t_0)A}x(t_0) + \int_{t_0}^{t} e^{(t-\tau)A}Bu(\tau)d\tau,\ t, t_0 \in \mathbb{R}, \\ y(t) &= D \circ u + Ce^{(t-t_0)A}x(t_0) + C\int_{t_0}^{t} e^{(t-\tau)A}Bu(\tau)d\tau. \end{aligned} \tag{1.31}$$

The outputs x of $\mathcal{B}_s$ and y of $\mathcal{B}$ for $t \geq t_0$ are thus determined by the input $u|_{[t_0,\infty)}$ for $t \geq t_0$ and the initial vector $x(t_0)$ at $t = t_0$. Therefore $x \in \mathcal{F}^n$ is called the *state* of $\mathcal{B}_s$ and of $\mathcal{B}$ and $x(t_0) \in \mathbb{C}^n$ the state at time t_0 . The isomorphism (1.28) is called a *state space representation or realization* of $\mathcal{B}$ and of H. Its existence is a slight variant of Kalman's famous *realization theorem.* The injectivity of (1.28) defines the *observability* of the equations $s \circ x = Ax + Bu$ and $y = Cx + D \circ u$. The isomorphism (1.28) is used, in particular, to (i) *simulate*, for $D \in \mathbb{C}^{p\times m}$, the trajectories of $\mathcal{B}$ by those of $\mathcal{B}_s$ and (ii) to derive the properties of the general IO behavior $\mathcal{B}$ from those of the state behavior $\mathcal{B}_s$, for instance in [3, pp. 560-], [21, Ch. 6] and [36, Ch. 7, p. 283]. In this book these applications do not play a dominant role. As in electrical engineering the most important results on IO behaviors, for instance of electrical and mechanical networks, with the equations $P \circ y = Q \circ u$ will be derived directly form $(P, -Q)$ and not from the state space representation (1.28) of the IO behavior.
The algorithmic computation of A, B, C, D is difficult, cf. [39, Ch. 6], [21, Ch. 6]. We compute four state space realizations of an IO behavior, usually called the *observability, observer, controllability resp. controller* realization, by means of the *Gröbner basis algorithm* in Chapter 11. These realizations depend only on the behavior, its IO structure and a chosen term order for the Gröbner theory, and are therefore called *canonical.* They give rise to the *observability* and *controllability indices* in connection with (1.28), cf. Theorem 11.3.2, [39, §6.4.6]. The ensuing algorithms are stronger and more general than those of [39, §6.4, §7.1], are directly implementable and are demonstrated in Examples 11.3.11, 11.4.9, 11.4.11, 11.4.15. The most important consequence of the observer realization is the so-called *pole shifting* algorithm, cf. Theorem 11.4.12, Corollary 11.4.14: If observability holds, i.e., if the map (1.28) is injective, and $f \in \mathbb{C}[s]$ is any monic polynomial of degree n, then the algorithm furnishes a matrix $L \in \mathbb{C}^{n\times p}$ such that $\det(s\,\mathrm{id}_n - (A - LC)) = f$.
If D is a nonconstant polynomial matrix, the vector $D \circ u$ may be distributional, for instance $s \circ Y = \delta$. Kalman avoided this in the following fashion: If $h = fg^{-1}$ is rational, one defines the s-degree of h as $\deg(h) := \deg_s(h) := \deg_s(f) - \deg_s(g)$, for instance $\deg(s^{-1}) = -1$, $\deg(0) := -\infty$. Then h is called *proper* resp. *strictly proper* if $\deg(h) \leq 0$ resp. $\deg(h) \leq -1$. Euclidean division of f by g furnishes a unique decomposition $h = h_{\mathrm{pol}} + h_{\mathrm{spr}}$ into a polynomial h_{pol} and a strictly proper h_{spr}, for instance $(s^2+1)(s+1)^{-1} = (s-1) + 2(s+1)^{-1}$. Then h is proper (strictly proper) if and only if $h_{\mathrm{pol}} \in \mathbb{C}$ ($h_{\mathrm{pol}} = 0$). The degree of $H = (H_{\nu\mu})_{\nu,\mu} \in \mathbb{C}(s)^{p\times m}$ is $\deg(H) := \deg_s(H) := \max_{\mu,\nu} \deg_s(H_{\nu\mu})$. The 'proper'-language and the decomposition $H = H_{\mathrm{pol}} + H_{\mathrm{spr}}$ are extended to matrices componentwise. Cramer's rule implies that $(s\,\mathrm{id}_n - A)^{-1}$ is strictly proper. Hence so is H_s, and $H = D + C(s\,\mathrm{id}_n - A)^{-1}B$ is the decomposition $H = H_{\mathrm{pol}} + H_{\mathrm{spr}}$. We infer that $D \in \mathbb{C}^{p\times m}$ if and only if H is proper. Equation (1.31) and its term $D \circ u$ then imply that H *is proper if and only if for all trajec-*

tories $\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right) \in \mathcal{B}$ *the piecewise continuity of* u *implies that of* y*, or, equivalently, if* $u \in \left(\mathrm{C}_+^{0,\mathrm{pc}}\right)^m$ *implies* $H \circ u \in \left(\mathrm{C}_+^{0,\mathrm{pc}}\right)^p$, cf. Theorem 5.3.1. Thus a proper transfer matrix induces the transfer operator $H\circ : \left(\mathrm{C}_+^{0,\mathrm{pc}}\right)^m \to \left(\mathrm{C}_+^{0,\mathrm{pc}}\right)^p$. The property of the input u can be chosen whereas that of y is determined by the behavior. Components of the behavior, that are distributions and not piecewise continuous, generally imply destruction or malfunctioning of a real system that is modelled by the behavior. In electrical engineering they say that the network burns out or saturates. Therefore, behaviors with nonproper transfer matrix have to be redesigned, for instance by choosing a different IO structure and ensuing transfer matrix, cf. [13, §2.5.3].
For proper H the behavior $\mathcal{B}_s$, the operator $\left(\begin{smallmatrix} C & D \\ 0 & \mathrm{id}_m \end{smallmatrix}\right)$ and thus H can be realized by interconnection of elementary building blocks, cf. Section 6.2. In this context one talks about the *synthesis and simulation* of H by means of $s \circ x = Ax + Bu,\ y = Cx + Du$.
Rosenbrock's equations generalize Kalman's state space equations in the form

$$\begin{aligned} &A \circ x = B \circ u,\ y = C \circ x + D \circ u \text{ with} \\ &A \in \mathbb{C}[s]^{n\times n},\ \operatorname{rank}(A) = n,\ B \in \mathbb{C}[s]^{n\times m},\ C \in \mathbb{C}[s]^{p\times n},\ D \in \mathbb{C}[s]^{p\times m}, \end{aligned} \tag{1.32}$$

cf. Theorem 5.3.1. These equations give rise to the behaviors

$$\begin{aligned} &\mathcal{B}_1 := \left\{\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in \mathcal{F}^{n+m};\ A \circ x = B \circ u\right\},\ H_1 := A^{-1}B, \\ &\mathcal{B}_2 := \left(\begin{smallmatrix} C & D \\ 0 & \mathrm{id}_m \end{smallmatrix}\right) \circ \mathcal{B}_1 = \left\{\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right) \in \mathcal{F}^{p+m};\ P \circ y = Q \circ u\right\} \text{ where} \\ &\qquad (P, -Q) \in \mathbb{C}[s]^{p\times(p+m)},\ \operatorname{rank}(P) = p,\ H_2 := P^{-1}Q = D + CA^{-1}B. \end{aligned} \tag{1.33}$$

It is obvious that $\mathcal{B}_1$ is an IO behavior with input u and transfer matrix H_1 and that $\mathcal{B}_2$ is a behavior as an image of $\mathcal{B}_1$. It turns out that $\mathcal{B}_2$ is also an IO behavior with input u and the indicated transfer matrix, and that the matrix $(P, -Q)$ can be computed from A, B, C, D. Here x is called the *pseudo-state*. In Willems' language the behavior $\mathcal{B}_2$ is obtained by eliminating the *latent variable* x from $\left\{\left(\begin{smallmatrix} x \\ y \\ u \end{smallmatrix}\right) \in \mathcal{F}^{n+p+m};\ A \circ x = B \circ u,\ y = C \circ x + D \circ u\right\} \cong \mathcal{B}_1$, $\left(\begin{smallmatrix} x \\ y \\ u \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right)$. Rosenbrock equations are the basic equations in [63], [75], [19], [72] and are intensively studied in [39], [21], [3], [13]. They appear at various places in this book, but are not predominant.
For the discussion below we also need the set of poles of H. Let $\mathrm{V}_{\mathbb{C}}(g) \subset \mathbb{C}$ denote the finite set of *roots or zeros* of a nonzero polynomial g. If $h = fg^{-1}$ is a rational function with coprime f and g, i.e. with greatest common divisor $\gcd(f, g) = 1$, we define the *set of poles resp. the domain* of h by $\operatorname{pole}(h) := \mathrm{V}_{\mathbb{C}}(g)$ resp. $\operatorname{dom}(h) := \mathbb{C} \setminus \operatorname{pole}(h)$. For $\lambda \in \operatorname{dom}(h)$ the value $h(\lambda) := f(\lambda)g(\lambda)^{-1} \in \mathbb{C}$ is defined, sometimes $h(\lambda) := \infty$ for $\lambda \in \operatorname{pole}(h)$ is used. The set of poles of the rational matrix H is $\operatorname{pole}(H) := \bigcup_{\beta,\alpha} \operatorname{pole}(H_{\beta\alpha})$, its complement is $\operatorname{dom}(H)$. For all $\lambda \in \operatorname{dom}(H)$ the matrix $H(\lambda) \in \mathbb{C}^{p\times m}$ is defined. The set $\operatorname{pole}(H)$ plays an important part in stability theory, see Section 1.6.

1.3 Distributions of finite order, impulse response and Laplace transform

We discuss the choice of the function space $\mathcal{F}$ again. The basic equations of LTI systems theory are the differential systems $P \circ y = Q \circ u$ from (1.11) where u, y have components in $\mathcal{F}$. The entries of P, Q are polynomials of arbitrarily high degree. This leads to the requirement that $\mathcal{F}$ be closed under differentiation or a $\mathbb{C}[s]$-module. Input signals $u = \alpha Y$, $\alpha \in \mathrm{t}(\mathcal{F})^m$, with a jump at $t = 0$ play an important part in electrical engineering, but have, in general, no derivative at $t = 0$ in the standard sense. This suggests to introduce a larger space $\mathcal{F}$ that includes all these and all continuous signals and their derivatives. This is similar to the extensions $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$. The famous solution of this problem is Schwartz' distribution theory [67], [37] and space $\mathcal{D}'$ of distributions. Let $\mathrm{C}_0^\infty := \mathrm{C}_0^\infty(\mathbb{R}, \mathbb{C})\ (\subset \mathrm{C}^\infty)$ denote the space of smooth functions φ with compact support, i.e., with $\varphi(t) = 0,\ |t| \geq r$, for some $r \geq 0$. With a suitable topology this is a topological vector space. Then $\mathcal{D}' \subset \mathrm{Hom}_{\mathbb{C}}(\mathrm{C}_0^\infty, \mathbb{C})$ is the space of *continuous* $\mathbb{C}$-linear functions from C_0^∞ to $\mathbb{C}$. The space $\mathrm{C}^{0,\mathrm{pc}}$ is embedded into $\mathcal{D}'$ via the monomorphism

$$\mathrm{C}^{0,\mathrm{pc}} \to \mathcal{D}',\ u \mapsto (\varphi \mapsto u(\varphi)),\ u(\varphi) := \int_{-\infty}^{\infty} u(t)\varphi(t)dt \Longrightarrow \mathrm{C}^{0,\mathrm{pc}} \underset{\text{identification}}{\subset} \mathcal{D}'. \tag{1.34}$$

Schwartz' theory in [67] is difficult, and so is Hörmander's very elegant form of it [37]. Since these theories do not belong to the mathematical knowledge of the first two university years, neither in mathematics nor in engineering, we proceed with a less elegant, but simpler method, cf. Section 7.2. We do not discuss the vector space topologies and replace (1.34) by the $\mathbb{C}$-monomorphism

$$\mathrm{C}^{0,\mathrm{pc}} \to \mathcal{D}^* := \mathrm{Hom}_{\mathbb{C}}(\mathrm{C}_0^\infty, \mathbb{C}),\ u \mapsto (\varphi \mapsto u(\varphi)),\ u(\varphi) := \int_{-\infty}^{\infty} u(t)\varphi(t)dt. \tag{1.35}$$

Again we identify $\mathrm{C}^{0,\mathrm{pc}} \subset \mathcal{D}^*$ via $u = (\varphi \mapsto u(\varphi))$. We make $\mathcal{D}^*$ a $\mathbb{C}[s]$-module by means of

$$(s \circ u)(\varphi) := u\left(-(s \circ \varphi)\right),\ u \in \mathcal{D}^*,\ \varphi \in \mathrm{C}_0^\infty. \tag{1.36}$$

The $-$-sign is chosen in order that $s \circ u = u'$ for a function $u \in \mathrm{C}^1$. In particular,

$$\begin{aligned} \delta := dY/dt,\ \delta(\varphi) &= -\int_{-\infty}^{\infty} Y(t)\varphi'(t)dt \\ &= -\int_0^{\infty} \varphi'(t)dt = -(\varphi(\infty) - \varphi(0)) = \varphi(0). \end{aligned} \tag{1.37}$$

Let $u_n \geq 0$ be a sequence of continuous functions with $u_n(t) = 0$ for $|t| \geq n^{-1}$ and $\int_{-1/n}^{1/n} u_n(t)dt = 1$ and hence $\lim_{n\to\infty} u_n(0) = \infty$. Then

$$\delta(\varphi) = \varphi(0) = \lim_{n\to\infty} u_n(\varphi), \tag{1.38}$$

cf. Theorem 7.2.10, and this suggested to call δ an *impulse at time* $t = 0$. Such an impulse is, of course, a mathematical idealization, as are all distributions that are not functions. Due to (1.38) a real system can be destroyed if certain

components are distributions. This has to be avoided by a better design. In many engineering books δ is suggestively introduced as $\delta \geq 0$, $\delta(t) := 0$ for $t \neq 0$ and $\int_{-\infty}^{\infty} \delta(t)\varphi(t)dt := \varphi(0)$. The distribution theory gives this definition a well-defined sense. Higher order derivatives of δ are needed, but are not introduced in the quoted textbooks. Distributions cannot be avoided by omitting the discontinuities of the signals [19, §3.2.1], for instance $s \circ Y = \delta$, but $s \circ Y|_{\mathbb{R}\setminus\{0\}} = 0$. If u is a Lebesgue absolutely integrable function on $\mathbb{R}$, trivially $\int_{0-}^{\infty} u(t)dt = \int_{0+}^{\infty} u(t)dt$. In particular, the Laplace transforms $\mathcal{L}_+$ and $\mathcal{L}_-$, defined by $\mathcal{L}_+(u)(s) = \int_{0+}^{\infty} u(t)e^{-st}dt$ and $\mathcal{L}_-(u)(s) = \int_{0-}^{\infty} u(t)e^{-st}dt$ for $s \in \mathbb{C}$ with $\Re(s) \geq 0$ coincide for such an u. These two integrals can differ only if u is a distribution with support in $[0, \infty)$ and the integral is properly redefined by means of distribution theory, cf. [39, §1.2], [13, pp. 381, 395].
The space $\mathcal{D}^*$ contains many elements that are of no analytic interest. Therefore, we only consider the $\mathbb{C}[s]$-submodule of $\mathcal{D}^*$, generated by C^0, i.e.,

$$\mathrm{C}^{-\infty} := \mathrm{C}^{-\infty}(\mathbb{R}, \mathbb{C}) = \bigcup_{n=0}^{\infty} s^n \circ \mathrm{C}^0 \subset \mathcal{D}' \subset \mathcal{D}^*. \tag{1.39}$$

This is the, now well-defined, $\mathbb{C}[s]$-module of all derivatives of all continuous functions, and is called the space of *distributions of finite order* [37, Thm. 4.4.7]. Many properties of $\mathcal{F} := \mathrm{C}^{-\infty}$ are first introduced for $\mathcal{D}^*$ by purely algebraic means and then carried over to $\mathrm{C}^{-\infty}$. We emphasize that this algebraic introduction of distributions works only in dimension one, i.e., for functions of one variable t. The $\mathbb{C}(s)$-vector spaces $\mathrm{C}_+^{-\infty}$ and $\mathcal{F}_2$ and the injective cogenerator $\mathcal{F}_4$ follow according to (1.14), (1.15), (1.18).
Since $\mathrm{C}_+^{-\infty}$ is a $\mathbb{C}(s)$-vector space the map $\mathbb{C}(s) \to \mathrm{C}_+^{-\infty}$, $H \mapsto h := H \circ \delta$, is injective. Since δ is interpreted as an impulse, h is called the *impulse response* of H. The *partial fraction decomposition* of H, cf. Section 4.5.2, then implies the $\mathbb{C}(s)$-isomorphism, cf. Theorem and Definition 7.2.47,

$$\begin{aligned} &\mathcal{L}^{-1} : \mathbb{C}(s) \cong \mathcal{F}_2 = \mathbb{C}[s] \circ \delta \bigoplus \left(\oplus_{\lambda \in \mathbb{C}} \mathbb{C}[t]e^{\lambda t}\right) Y,\ H \mapsto H \circ \delta, \text{ with} \\ &\forall k \geq 0 :\ \mathcal{L}^{-1}(s^k) = \delta^{(k)},\ \forall k \geq 1,\ \lambda \in \mathbb{C} :\ \mathcal{L}^{-1}((s-\lambda)^{-k}) = \tfrac{t^{k-1}}{(k-1)!} e^{\lambda t} Y. \end{aligned} \tag{1.40}$$

The map $\mathcal{L}^{-1}$ (an image $\mathcal{L}^{-1}(H)$) is called the *inverse Laplace transform* (of H), and its inverse $\mathcal{L}$ (image $\mathcal{L}(h)$) the *Laplace transform* (of h). There results the bijective correspondence

$$\mathcal{F}_2 \ni h = \mathcal{L}^{-1}(H) = H \circ \delta \longleftrightarrow H = \mathcal{L}(h) \in \mathbb{C}(s). \tag{1.41}$$

The constructive form of $\mathcal{L}$ follows directly from (1.40). In *electrical engineering* tables are in use to compute $\mathcal{L}^{-1}(H)$ and $\mathcal{L}(h)$ in special cases, c.f. [2, pp. 253-256]. The constructive partial fraction decomposition furnishes these computations for all H and h. There are the additional equivalences

$$\begin{aligned} &h := H \circ \delta = \alpha Y,\ \alpha \in \mathrm{t}(\mathcal{F}) \Longleftrightarrow H = \mathcal{L}(h) \text{ strictly proper or } \deg_s(H) \leq -1, \\ &h = H \circ \delta \text{ continuous} \Longleftrightarrow sH \text{ strictly proper or } \deg_s(H) \leq -2 \Longleftrightarrow \alpha(0) = 0. \end{aligned} \tag{1.42}$$

The maps $\mathcal{L}$ and $\mathcal{L}^{-1}$ are extended to matrices componentwise such that (1.40) and (1.42) hold likewise for matrices. Assume the IO behavior from (1.11) with

its transfer matrix H and its impulse response $h := H \circ \delta = \mathcal{L}^{-1}(H),\ \mathcal{L}(h) = H$. Notice that we derived the transfer matrix H of an IO behavior by module-behavior duality, whereas in almost all engineering books H is *defined* by the equation $H = \mathcal{L}(h)$. This definition depends on the special matrices, defining the behavior, and requires, of course, that $\mathcal{L}$ and h have been defined before. For this approach, that is mostly not carried out with all exact mathematical details, we refer to the quoted textbooks, especially [70, §6.6] in electrical engineering, and to (1.48)-(1.52) below.
Consider any input signal $u = H_2 \circ \delta = \mathcal{L}^{-1}(H_2),\ H_2 \in \mathbb{C}(s)^m$. These signals, in particular $u = H_2 \circ \delta = \alpha Y,\ \alpha \in \mathrm{t}(\mathcal{F})^m$, for strictly proper H_2, are by far the most important ones with left bounded support in electrical engineering, cf. [2, §5.2]. They occur if a electrical network is switched on at time $t = 0$. Then all outputs of $\mathcal{B}$ to the input u have the form, cf. Theorem 7.2.53,

$$\begin{aligned} &y = y_{ss} + z,\ y_{ss} := HH_2 \circ \delta = \mathcal{L}^{-1}(HH_2),\ z \in \mathcal{B}^0 \subset \mathrm{t}(\mathcal{F})^p \\ &\Longrightarrow P \circ y_{ss} = PHH_2 \circ \delta = QH_2 \circ \delta = Q \circ (H_2 \circ \delta) = Q \circ u, \\ &\quad P\mathcal{L}(y_{ss}) = PHH_2 = QH_2 = Q\mathcal{L}(u),\ \mathcal{L}(y_{ss}) = H\mathcal{L}(u). \end{aligned} \tag{1.43}$$

Here y_{ss} can be easily computed with the partial fraction decomposition of HH_2. If $\mathcal{B}$ is asymptotically stable, cf. Section 1.6, y_{ss} resp. z are again called the *steady or stationary state resp. the transient* of y.
In Theorem 7.2.53 we assume that HH_2 is strictly proper with $HH_2 \circ \delta = \beta Y,\ \beta \in \mathrm{t}(\mathcal{F})^p$, to compute the unique solution of the *initial value problem*, cf. [70, §6.6],

$$\begin{aligned} &P \circ y = Q \circ (H_2 \circ \delta) \text{ with given } y_k^{(\mu)}(0+) := \lim_{t \to 0, t > 0} y_k^{(\mu)}(t) \in \mathbb{C} \text{ for} \\ &(k, \mu) \in \Gamma^{\mathrm{ob}} := \left\{(k, \mu); 1 \le k \le p,\ 0 \le \mu \le d^{\mathrm{ob}}(k) - 1\right\} \text{ as} \\ &y = \beta Y + C^{\mathrm{ob}} e^{tA^{\mathrm{ob}}} \left(y_k^{(\mu)}(0+) - \beta_k^{(\mu)}(0)\right)_{(k,\mu) \in \Gamma^{\mathrm{ob}}} \text{ where} \\ &A^{\mathrm{ob}} \in F^{\Gamma^{\mathrm{ob}} \times \Gamma^{\mathrm{ob}}},\ C^{\mathrm{ob}} \in F^{p \times \Gamma^{\mathrm{ob}}},\ C^{\mathrm{ob}}_{i,(k,\mu)} = \delta_{i,k}\delta_{0,\mu} \text{ if } d^{\mathrm{ob}}(i) > 0. \end{aligned} \tag{1.44}$$

The *observability indices* $d^{\mathrm{ob}}(k) \ge 0,\ k = 1, \cdots, p$, and the matrices $A^{\mathrm{ob}},\ C^{\mathrm{ob}}$ of the canonical observability realization of $\mathcal{B}$ are introduced and computed in Section 11.3. For instance [39, p. 11, Ex. 1.2-1.],

$$y' + 2y = \delta \text{ or } (s+2) \circ y = 1 \circ \delta,\ y(0+) := 2,\ (s+2)^{-1} \circ \delta = e^{-2t}Y,\ y = e^{-2t}(Y+1).$$

According to (1.44) the transient $z = y - \beta Y$ can be determined precisely if and only if the initial values $y_k^{(\mu)}(0+)$ are known precisely from exact measurements. Measuring devices for high order derivatives of signals do not exist in general. Hence, in general, the transient is not known precisely, but the matrix $C^{\mathrm{ob}} e^{tA^{\mathrm{ob}}}$ in (1.44) determines the general form of its decay. This remark applies to most transients discussed in this book. For low order derivatives such measuring devices exist, for instance speedometers and accelerometers, and only for such cases examples can be found in the quoted textbooks.
For most practical signals u, (1.43) is the best method to compute $H \circ u$. For certain *proofs*, however, the representation as *convolution* is needed, cf. Section 7.2.5. The convolution of two continuous functions u_1 and u_2 with $u_i(t) = 0$ for $t \le t_i,\ i = 1, 2$, is the continuous function $(u_1 * u_2)(t) := \int_{-\infty}^{\infty} u_1(t - \tau) u_2(\tau) d\tau$.

This integral is indeed a finite Riemann integral, no Lebesgue theory is needed or used. The convolution is commutative and associative, and is uniquely extended to all derivatives of continuous functions with left bounded support, i.e., to the convolution product $\mathrm{C}_+^{-\infty} \times \mathrm{C}_+^{-\infty} \to \mathrm{C}_+^{-\infty}$, $(u_1, u_2) \mapsto u_1 * u_2$. This makes $\mathrm{C}_+^{-\infty}$ a commutative $\mathbb{C}$-algebra with the 1-element δ, i.e., $\delta * u = u$, and the rule $H \circ (u_1 * u_2) = (H \circ u_1) * u_2$, $H \in \mathbb{C}(s)$. As usual, the convolution is extended to matrices componentwise. We infer

$$\begin{aligned} &H \circ u = H \circ (\delta * u) = (H \circ \delta) * u = h * u,\ h := H \circ \delta, \text{ where} \\ &H = P^{-1}Q \in \mathbb{C}(s)^{p\times m},\ h \in \mathcal{F}_2^{p\times m} \subset \left(\mathrm{C}_+^{-\infty}\right)^{p\times m},\ u \in \left(\mathrm{C}_+^{-\infty}\right)^m. \end{aligned} \tag{1.45}$$

Due to $P \circ h = Q \circ \delta$ and $P \circ (h * u) = Q \circ u$ the impulse response matrix $h = H \circ \delta$ is also called the *fundamental solution* of $P \circ y = Q \circ u$ [37, p. 80]. If

$$\begin{aligned} H = H_{\mathrm{pol}} + H_{\mathrm{spr}},\ H_{\mathrm{pol}} = \sum_{k=0}^{d} H_k s^k \in \mathbb{C}[s]^{p\times m},\ H_{\mathrm{spr}} \circ \delta = \alpha Y,\ u \in \left(\mathrm{C}_+^{0,\mathrm{pc}}\right)^m \\ \text{with } H_k \in \mathbb{C}^{p\times m},\ \alpha \in \mathrm{t}(\mathcal{F})^{p\times m}, \text{ then} \\ H \circ u = H_{\mathrm{pol}} \circ u + H_{\mathrm{spr}} \circ u = \sum_{k=0}^{d} H_k s^k \circ u + \int_{-\infty}^{t} \alpha(t-\tau)u(\tau)d\tau \end{aligned} \tag{1.46}$$

where the integral is Riemann, finite and continuous in t, cf. [19, p. 95]. If $H_{\mathrm{pol}} = H_0$ or H is proper, then $H \circ u = H_0 u + H_{\mathrm{spr}} \circ u \in \left(\mathrm{C}_+^{0,\mathrm{pc}}\right)^p$. The equation $H_1 H_2 \circ \delta = H_1 H_2 \circ (\delta * \delta) = (H_1 \circ \delta) * (H_2 \circ \delta)$ implies that $\mathcal{F}_2 = \mathbb{C}[s] \circ \delta \oplus \mathrm{t}(\mathcal{F})Y$ is a subalgebra of $\mathrm{C}_+^{-\infty}$ and that the Laplace transform and its inverse are algebra isomorphisms. Since $\mathbb{C}(s)$ is a field, so is $\mathcal{F}_2$.
By reduction to the case

$$H_2 = (s-\lambda)^{-k},\ \lambda \in \mathbb{C},\ k \geq 1,\ H_2 \circ \delta = \tfrac{t^{k-1}}{(k-1)!} e^{\lambda t} Y$$

one shows

$$\begin{aligned} H_2(s) = \mathcal{L}(H_2 \circ \delta)(s) = \mathcal{L}(\alpha Y)(s) = \int_0^\infty \alpha(t) e^{-st} dt \text{ for } s \in \{z \in \mathbb{C};\ \Re(z) > \sigma\} \\ \text{if } \deg_s(H_2) \leq -1,\ H_2 \circ \delta = \alpha Y,\ \alpha \in \mathrm{t}(\mathcal{F})^m,\ \sigma \geq \max\{\Re(\lambda);\ \lambda \in \mathrm{pole}(H_2)\}. \end{aligned} \tag{1.47}$$

This is the standard engineering definition of the Laplace transform of αY and suggested to extend the definition of $\mathcal{L}$ to more general distributions in the following fashion, cf. Theorem 7.2.87 in Section 7.2.7 and Theorem 9.2.51: A function $u \in \mathrm{C}_+^{0,\mathrm{pc}}$ is called *Laplace transformable* if there is $\sigma > 0$ such that $|u(t)|e^{-\sigma t}$, $t \in \mathbb{R}$, is bounded. Then the function $u(t)e^{-st}$ for $s \in \mathbb{C}$, $\Re(s) > \sigma$, is absolutely integrable on $\mathbb{R}$ and

$$\mathcal{L}(u)(s) := \int_{-\infty}^{\infty} u(t) e^{-st} dt, s \in \{z \in \mathbb{C};\ \Re(z) > \sigma\}, \tag{1.48}$$

is a holomorphic function of s in the open half-plane $\{z \in \mathbb{C};\ \Re(z) > \sigma\}$. The higher derivatives $s^n \circ v$, $n \geq 0$, of Laplace transformable functions v are called Laplace transformable distributions. They form the subset $\mathfrak{A}_+ \subset \mathrm{C}_+^{-\infty}$. For

such an $u = s^n \circ v \in \mathfrak{A}_+$ one defines the *Laplace transform* $\mathcal{L}(u)$ of u as the holomorphic function $\mathcal{L}(u)(s) := s^n \mathcal{L}(v)(s)$, $\Re(s) > \sigma$. As usual $\mathcal{L}$ is extended to matrices componentwise. Then $\mathfrak{A}_+$ and $\mathcal{L}$ have the following properties: The set $\mathfrak{A}_+$ is a $\mathbb{C}(s)$-subspace of $\mathrm{C}_+^{-\infty}$ and $\mathcal{L}$ is $\mathbb{C}(s)$-linear on $\mathfrak{A}_+$, i.e.,

$$\mathcal{L}(H \circ u)(s) = H(s)\mathcal{L}(u)(s),\ H \in \mathbb{C}(s),\ u \in \mathfrak{A}_+,\ s \in \{z \in \mathbb{C};\ \Re(z) > \sigma\}, \tag{1.49}$$

for some $\sigma > 0$, depending on H and u. The $\mathbb{C}(s)$-subspace $\mathcal{F}_2 = \mathbb{C}[s] \circ \delta \bigoplus \oplus_{\lambda \in \mathbb{C}} \mathbb{C}[t]e^{\lambda t} \subset \mathrm{C}_+^{-\infty}$ is contained in $\mathfrak{A}_+$ and $\mathcal{L}$ extends $\mathcal{L}$ from Theorem 7.2.47, i.e., $\mathcal{L}(H \circ \delta) = H$. The map $\mathcal{L}$ is injective, i.e., $\mathcal{L}(u) = 0$ implies $u = 0$. If $u \in \mathrm{C}_+^0$ is Laplace tranformable with $\sigma > 0$, if $\rho > \sigma$ and if $\mathcal{L}(u)(\rho + j\omega)$ is absolutely integrable as function of ω, then the following inversion formula holds, cf. [39, §1.2, (2),(4)], [70, (6.114)], [20, p. 485,(12)], [13, p. 398, (12.56)]:

$$u(t) = (2\pi j)^{-1} \int_{\rho - j\infty}^{\rho + j\infty} \mathcal{L}(u)(s) e^{st} ds,\ j := \sqrt{-1}. \tag{1.50}$$

Finally $\mathfrak{A}_+$ satisfies the *exchange theorem*, i.e., $\mathfrak{A}_+$ is a unital subalgebra of the convolution algebra $\left(\mathrm{C}_+^{-\infty}, *\right)$ with one-element δ, $\mathcal{L}(\delta) = 1$ and $\mathcal{L}$ is multiplicative on $\mathfrak{A}_+$, i.e.,

$$\mathcal{L}(u_1 * u_2)(s) = \mathcal{L}(u_1)(s)\mathcal{L}(u_2)(s),\ u_1, u_2 \in \mathfrak{A}_+,\ s \in \{z \in \mathbb{C};\ \Re(z) > \sigma\} \tag{1.51}$$

for some $\sigma > 0$, depending on u_1 and u_2. If, in particular, $u \in \mathfrak{A}_+^m \subset \left(\mathrm{C}_+^{-\infty}\right)^m$ is a Laplace transformable input of the IO behavior $\mathcal{B}$ from (1.11), then the unique output $y := H \circ u \in \left(\mathrm{C}_+^{-\infty}\right)^p$ with $P \circ y = Q \circ u$ is also Laplace transformable and

$$\mathcal{L}(y)(s) = H(s)\mathcal{L}(u)(s),\ P(s)\mathcal{L}(y)(s) = Q(s)\mathcal{L}(u)(s),\ s \in \{z \in \mathbb{C};\ \Re(z) > \sigma\}, \tag{1.52}$$

for some $\sigma > 0$. The latter equation holds without the usually required *zero initial conditions* [39, p. 551], [19, p. 94, (21)], [72, p. 55], [13, Thm. 24 on p. 37] or *relaxedness assumptions* [21, p. 82]. The equations (1.49)-(1.52) are not proven in detail in the quoted textbooks, but are there essential for the *definition* of the transfer matrix H. Our proof in Theorem 7.2.89 is short and elementary and, in particular, does not use the Fourier transform of temperate distributions, cf. [13, §12.3.4]. In the present book the Laplace transform on $\mathcal{F}_2$ from (1.40) and from Theorem 7.2.47 suffices for all considered applications.

For a Laplace transformable distribution u with support in $[0, \infty)$ its Laplace transform, as already mentioned, is often *defined* [39, p. 10], [20, (5) on p. 482], [13, §12.3.4, (12.48)] as

$$\mathcal{L}(u)(s) = \int_{0-}^{\infty} u(t) e^{-st} dt.$$

Unless u is a Laplace transformable function, this expression and its further use require a precise distributional explanation. For a smooth function u the equation $(uY)' = u'Y + u(0)\delta$ implies $\mathcal{L}(u'Y) = s\mathcal{L}(uY) - u(0)$ [13, p. 396]. It is often written as $\mathcal{L}(u') = s\mathcal{L}(u) - u(0)$ [20, p. 185], [3, p. 155] and then seems to contradict the rule $\mathcal{L}(s \circ u) = s\mathcal{L}(u)$.
Many authors, e.g. [60, §2.3.2], use the space $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C})$ as the basic signal space

and a different notion of *weak* solution of a differential equation. In our opinion these are inappropriate for the following reasons: This space is a factor space $\mathrm{L}^1_{\mathrm{loc}} = \mathcal{L}^1_{\mathrm{loc}}/\mathcal{L}_0$. A function w in $\mathcal{L}^1_{\mathrm{loc}}$ is a Lebesgue measurable function whose Lebesgue integrals $\int_a^b |w(t)|dt,\ a,b \in \mathbb{R}, a < b$, are finite whereas a function in $\mathcal{L}_0$ is measurable and zero almost everywhere. An element of $\mathrm{L}^1_{\mathrm{loc}}$ is a residue class $\overline{w} := w + \mathcal{L}_0,\ w \in \mathcal{L}^1_{\mathrm{loc}}$, and hence $\overline{w}(t),\ t \in \mathbb{R}$, is not defined, i.e., $\overline{w}$ has no functional values. In contrast to piecewise continuous signals such signals can neither be measured nor generated, a basic requirement for signals in electrical engineering. If $\mathcal{L}^1_{\mathrm{loc}}$ instead of $\mathrm{L}^1_{\mathrm{loc}}$ is used, then the basic implication $\left(\int_a^b |w(t)|dt = 0 \Longrightarrow w|_{[a,b]} = 0\right)$ does not hold. For $w_1, w_2 \in \mathrm{L}^1_{\mathrm{loc}}$ the following implications hold, cf. [67, Thm. III, p. 54]:

$$\begin{aligned} &s \circ w_1 = w_2 \text{ in } \mathrm{C}^{-\infty} \Longleftrightarrow \forall \varphi \in \mathrm{C}_0^\infty : -\int_{-\infty}^{\infty} w_1(x)\varphi'(x)dx = \int_{-\infty}^{\infty} w_2(x)\varphi(x)dx \\ &\Longrightarrow \exists c \in \mathbb{C} \text{ with } w_1 = \int_0^t w_2(x)dx + c \in \mathrm{L}^1_{\mathrm{loc}}. \end{aligned} \tag{1.53}$$

Moreover w_1 is then continuous and its usual derivative $w_1'(t)$ exists almost everywhere and coincides with w_2 in $\mathrm{L}^1_{\mathrm{loc}}$. The converse implication in the second line holds if w_1 is differentiable almost everywhere in the usual sense and $w_2 = w_1' \in \mathrm{L}^1_{\mathrm{loc}}$, but not in general. So a *weak* solution of $dw_1/dt = w_2$ according to [60, Def. 2.3.7], i.e., $w_1 = \int_0^t w_2(x)dx + c \in \mathrm{L}^1_{\mathrm{loc}}$, does *not* imply $s \circ w_1 = w_2$ in $\mathrm{C}^{-\infty}$. Also all piecewise continuous functions belong to $\mathrm{L}^1_{\mathrm{loc}}$, for instance Y, but their derivatives like $\delta = s \circ Y$ do not.

1.4 Periodic signals and Fourier series

Another important class of signals are the periodic ones, and Fourier series are an essential technical tool for these, cf. [2, Ch. 3] and Section 7.2.8. We assume the IO behavior from (1.11) with transfer matrix H. Let $T > 0$ and $\omega := 2\pi T^{-1}$. A piecewise continuous signal u is called T-periodic if $u(t) = u(t+T)$ for all $t \in \mathbb{R}$, the *sinusoidal* or *harmonic* functions $e^{j\mu\omega t},\ \mu \in \mathbb{Z},\ j := \sqrt{-1}$, being the standard examples. Let $\mathcal{P}^0$ ($\mathcal{P}^{0,\mathrm{pc}}$) be the space of (piecewise) continuous, T-periodic signals. The space $\mathcal{P}^{0,\mathrm{pc}}$ has the *inner product* $\langle u_1, u_2\rangle := T^{-1}\int_0^T \overline{u_1(t)}u_2(t)dt$ and the *induced norm* $\|u\|_2 := \langle u, u\rangle^{1/2}$ with $\|1\|_2 = 1$. The $e^{j\mu\omega t},\ \mu \in \mathbb{Z}$, form the standard orthonormal family of functions. For $u \in \mathcal{P}^{0,\mathrm{pc}}$ ones defines the *sequence of Fourier coefficients*

$$\begin{aligned} &\mathbb{F}(u) \in \mathbb{C}^{\mathbb{Z}} \text{ by } \mathbb{F}(u)(\mu) := \langle e^{j\mu\omega t}, u\rangle = T^{-1}\int_0^T e^{-j\mu\omega t}u(t)dt. \text{ Then} \\ &u = \sum_{\mu\in\mathbb{Z}} \mathbb{F}(u)(\mu)e^{j\mu\omega t}, \text{ i.e., } \lim_{N\to\infty} \|u - \sum_{\mu=-N}^{N} \mathbb{F}(u)(\mu)e^{j\mu\omega t}\|_2 = 0. \end{aligned} \tag{1.54}$$

The map $\mathbb{F}$ becomes a bijective transformation in the following fashion, cf. [67, §VII.1]. Like $\mathrm{C}^{-\infty}$ we define the subspace $\mathcal{P}^{-\infty}$ ($\subset \mathrm{C}^{-\infty}$) of periodic distributions as the space of derivatives $s^n \circ u,\ u \in \mathcal{P}^0,\ n \geq 0$. Again, no topological vector spaces are used. We define the sequence space $\mathfrak{s}^{-\infty} \subset \mathbb{C}^{\mathbb{Z}}$ of all sequences

$\widehat{u} \in \mathbb{C}^{\mathbb{Z}}$ that *grow at most polynomially*, i.e., for which there are $M > 0$ and $k \in \mathbb{Z}$ such that $|\widehat{u}(\mu)| \leq M(1+\mu^2)^k$ for all $\mu \in \mathbb{Z}$. Let $\mathbb{C}(s)_{\text{per}}$ denote the subalgebra of $\mathbb{C}(s)$ of all rational functions H without poles in $\mathbb{Z}j\omega$, i.e., for which $H(j\mu\omega) \in \mathbb{C}$ is defined for all $\mu \in \mathbb{Z}$. The space $\mathfrak{s}^{-\infty}$ becomes a $\mathbb{C}(s)_{\text{per}}$-module with the scalar multiplication $H \circ \widehat{u}$, defined by

$$(H \circ \widehat{u})(\mu) := H(j\mu\omega)\widehat{u}(\mu),\ H \in \mathbb{C}(s)_{\text{per}},\ \widehat{u} \in \mathfrak{s}^{-\infty},\ \mu \in \mathbb{Z}. \tag{1.55}$$

With these data the map $\mathbb{F}$ can be uniquely extended to a $\mathbb{C}(s)_{\text{per}}$-isomorphism

$$\mathbb{F}:\ \mathcal{P}^{-\infty} \cong \mathfrak{s}^{-\infty}, (\text{cf. Theorem 7.2.99})\,. \tag{1.56}$$

With respect to a suitable topology on $\mathcal{P}^{-\infty}$ [67, (VII,1;3)] that we, however, do not discuss, one obtains the convergent series $u = \sum_{\mu\in\mathbb{Z}} \mathbb{F}(u)(\mu)e^{j\mu\omega t}$ for $u \in \mathcal{P}^{-\infty}$. If $u \in \mathcal{P}^{0,\text{pc}}$ and if $\sum_{\mu\in\mathbb{Z}} |\mathbb{F}(u)(\mu)| < \infty$, then $u = \sum_{\mu\in\mathbb{Z}} \mathbb{F}(u)(\mu)e^{j\mu\omega t}$ is uniformly convergent and thus continuous.
As usual, $\mathbb{F}$ is extended to matrices componentwise. Assume the IO behavior from (1.11) and $H \in \mathbb{C}(s)^{p\times m}_{\text{per}}$. Then

$$H\circ:\ \left(\mathcal{P}^{-\infty}\right)^m \to \left(\mathcal{P}^{-\infty}\right)^p,\ u \mapsto H \circ u = \mathbb{F}^{-1}\left(H \circ \mathbb{F}(u)\right), \tag{1.57}$$

is another well-defined *transfer operator* such that for all $u \in \left(\mathcal{P}^{-\infty}\right)^m$ the trajectory $\left(\begin{smallmatrix} H\circ u \\ u \end{smallmatrix}\right)$ is periodic and belongs to $\mathcal{B}$. If, in addition, $P^{-1} \in \mathbb{C}(s)^{p\times p}_{\text{per}}$ or $\mathrm{V}_{\mathbb{C}}(\det(P)) \in \mathbb{C} \setminus \mathbb{Z}j\omega$, then H induces the $\mathbb{C}(s)_{\text{per}}$-isomorphism, c.f. (1.25),

$$\left(\mathcal{P}^{-\infty}\right)^m \cong \mathcal{B} \bigcap \left(\mathcal{P}^{-\infty}\right)^{p+m},\ u \mapsto \left(\begin{smallmatrix} H\circ u \\ u \end{smallmatrix}\right). \tag{1.58}$$

The obvious equations $\mathrm{C}^0_+ \cap \mathcal{P}^0 = 0$ and $\mathrm{C}^{-\infty}_+ \cap \mathcal{P}^{-\infty} = 0$ show that the maps $H\circ$ from (1.24) and (1.57) are independent of each other, but both satisfy $P \circ (H \circ u) = Q \circ u$, i.e., $\left(\begin{smallmatrix} H\circ u \\ u \end{smallmatrix}\right) \in \mathcal{B}$.
Assume additionally that $H = H_0 + H_{\text{spr}}$ is proper, i.e., $H_0 \in \mathbb{C}^{p\times m}$, and $u \in \left(\mathcal{P}^{0,\text{pc}}\right)^m$. Then the output signal $y_{ss} := H \circ u$ is, cf. Theorem 7.2.102,

$$y_{ss} = H \circ u = \sum_{\mu\in\mathbb{Z}} H(j\mu\omega)\mathbb{F}(u)(\mu)e^{j\mu\omega t} = H_0 u + \sum_{\mu\in\mathbb{Z}} H_{\text{spr}}(j\mu\omega)\mathbb{F}(u)(\mu)e^{j\mu\omega t} \tag{1.59}$$

where the second sum $\sum_\mu$ is uniformly convergent and thus continuous. Equation 1.59 is the most general form of the *superposition principle* for periodic signals, and an important tool for the analysis of electrical networks [70, §5.5.2], [2, §3.3.1]. If u is a *sinusoidal or harmonic* input of the simple form $u = u(0)e^{j\omega t}$, $u(0) \in \mathbb{C}^m$, $\omega > 0$, then (1.59) simplifies to

$$y_{ss} = H \circ u = H(j\omega)u = H(j\omega)u(0)e^{j\omega t}. \tag{1.60}$$

If $(P, -Q) \in \mathbb{R}[s]^{p\times(p+m)}$ and thus $H \in \mathbb{R}(s)^{p\times m}$ are real, we obtain

$$\begin{aligned}
\Re(u) &= \Re(u(0))\cos(\omega t) - \Im(u(0))\sin(\omega t),\\
\Im(u) &= \Im(u(0))\cos(\omega t) + \Re(u(0))\sin(\omega t),\\
\Re(y_{ss}) &= H \circ \Re(u) = \Re(H \circ u) = \Re\left((H(j\omega)u(0))e^{j\omega t}\right)\\
&= (\Re(H(j\omega))\Re(u(0)) - \Im(H(j\omega))\Im(u(0)))\cos(\omega t)\\
&\quad - (\Re(H(j\omega))\Im(u(0)) + \Im(H(j\omega))\Re(u(0)))\sin(\omega t).
\end{aligned} \tag{1.61}$$

The simplicity of (1.60) compared to (1.61) suggested the *complex method for real harmonic voltages and currents* in electrical engineering.
Notice again that for $\binom{y}{u} \in \left(\mathcal{P}^{0,\mathrm{pc}}\right)^{p+m}$ the basic equation $P \circ y = Q \circ u$ makes, in general, no sense without the space $\mathcal{P}^{-\infty}$ of periodic distributions that contains all derivatives of functions in $\mathcal{P}^{0,\mathrm{pc}}$. If $\mathcal{B}$ is asymptotically stable, cf. (1.102), and y is any output to u, i.e., solves $P \circ y = Q \circ u$, then y_{ss} resp. $y - y_{ss} \in \mathcal{B}^0$ are again called the *steady or stationary state resp. the transient of* y.
All practical signals in $\mathcal{P}^{0,\mathrm{pc}}$ are derived from polynomial-exponential functions, for instance the T-periodic signals

$$u_1(t) = 2T^{-1}t,\ -T/2 \le t < T/2,\ \text{or}\ u_2(t) := \begin{cases} 2T^{-1}t & \text{if } 0 \le t \le T/2 \\ 2T^{-1}(T-t) & \text{if } T/2 \le t \le T \end{cases}. \tag{1.62}$$

For these signals, (1.59) can be constructively improved, cf. Theorems 7.2.107 and 7.2.110.

1.5 Generalized fractional calculus and behaviors

With the same methods as for the general Laplace transform in Section 1.3 we study *fractional or symbolic calculus*, cf. [67, §VI,5], [Wikipedia; https://en.wikipedia.org/wiki/Fractional calculus, 4 September 2019], [43] and *fractional behaviors* in the last Chapter 12 , but we do not discuss the role of these in applications that are comprehensively treated in [43]. A suitable vector space-behavior duality is again the key to compute the trajectories of fractional behaviors, cf. Theorem 12.1.3. The partial fraction decomposition of rational matrices enables the constructive solution of very general linear systems of fractional integral/differential equations.
If f is a piecewise continuous, complex valued function on the open interval $(0, \infty)$ we extend this to a function $f_{\mathbb{R}} : \mathbb{R} \to \mathbb{C}$ by

$$\forall t > 0 :\ f_{\mathbb{R}}(t) := f(t),\ \forall t \le 0 :\ f_{\mathbb{R}}(t) := 0. \tag{1.63}$$

Obviously $f_{\mathbb{R}}$ is piecewise continuous on $\mathbb{R} \setminus \{0\}$. If $f(0+) := \lim_{t \to 0,\, t>0} f(t)$ exists, then $f_{\mathbb{R}}$ has the jump $f_{\mathbb{R}}(0+) - f_{\mathbb{R}}(0-) = f(0+)$ at $t = 0$ and is also piecewise continuous. If for some $a > 0$ the Riemann integral

$$\int_{-a}^{a} |f_{\mathbb{R}}(t)| dt = \int_0^a |f(t)| dt := \lim_{\epsilon \to 0,\, \epsilon > 0} \int_{\epsilon}^{a} |f(t)| dt < \infty$$

is finite, then $f_{\mathbb{R}}$ is called *locally integrable*. The corresponding distribution is defined by

$$\begin{aligned} &f_{\mathbb{R}}(\varphi) := \int_{-\infty}^{\infty} f_{\mathbb{R}}(t)\varphi(t)dt = \lim_{\epsilon \to 0,\, \epsilon > 0} \int_{\epsilon}^{\infty} f(t)\varphi(t)dt,\ \varphi \in \mathrm{C}_0^{\infty}, \Longrightarrow \forall u \in \mathrm{C}_+^{0,\mathrm{pc}} : \\ &(f_{\mathbb{R}} * u)(t) = \int_{-\infty}^{t} f(t-x)u(x)dx = \lim_{\epsilon \to 0, \epsilon > 0} \int_{-\infty}^{t-\epsilon} f(t-x)u(x)dx. \end{aligned} \tag{1.64}$$

If $f_{\mathbb{R}}$ is locally integrable and $u \in \mathrm{C}_+^{0,\mathrm{pc}}$, then $f_{\mathbb{R}} * u$ is continuous, i.e., $f_{\mathbb{R}} * u \in \mathrm{C}_+^0$. Since $f_{\mathbb{R}} = (s \circ Y) * f_{\mathbb{R}} = s \circ (Y * f_{\mathbb{R}})$ is the derivative of the continuous function $Y * f_{\mathbb{R}}$, $f_{\mathbb{R}}$ is a distribution of finite order and belongs to $\mathrm{C}_+^{-\infty}$.

Let $\Gamma(m),\ m \in \mathbb{C}$, denote the meromorphic *Gamma* function with $\Gamma(m+1) = m!$ for $m \in \mathbb{N}$. According to [67, §VI,5] one defines the *fractional integral operators*

$$\begin{aligned} &\forall m \in \mathbb{C}:\ I^m : \mathrm{C}_+^{-\infty} \to \mathrm{C}_+^{-\infty},\ y \mapsto Y_m * y,\ \text{where} \\ &Y_m := \begin{cases} \left(\Gamma(m)^{-1} t^{m-1}\right)_{\mathbb{R}} \in \mathrm{C}_+^0 & \text{if } \Re(m) - 1 > 0 \\ s^k \circ Y_{m+k} \in \mathrm{C}_+^{-\infty} & \text{if } k \in \mathbb{N} \text{ and } \Re(m) + k - 1 > 0 \end{cases}. \end{aligned} \tag{1.65}$$

Especially this implies

$$\begin{aligned} &\forall m \in \mathbb{Z}:\ Y_m = s^{-m} \circ \delta,\ \forall m \in \mathbb{N}:\ Y_{-m} = \delta^{(m)},\ Y_0 = \delta,\ Y_1 = Y, \\ &\forall m \in \mathbb{C} \forall u \in \mathrm{C}_+^{0,\mathrm{pc}}:\ I^m u = s^k \circ \Gamma(m+k)^{-1} \int_{-\infty}^{t} (t-x)^{m+k-1} u(x) dx. \end{aligned} \tag{1.66}$$

The general definition is independent of the choice of $k \in \mathbb{N}$ with $\Re(m)+k-1 > 0$. We note that $0 < \Re(m) + k \leq 1$ would suffice, but then Y_{m+k} is only locally integrable, but not continuous on $\mathbb{R}$. The equations

$$Y_1 = s^{-1} \circ \delta = Y \text{ and } I^1 u = Y_1 * u = Y * u = \int_{-\infty}^{t} u(t) dt,\ u \in \mathrm{C}_+^{0,\mathrm{pc}}, \tag{1.67}$$

suggested the notation Y_m for a generalized Heaviside function and to call I^m an integral operator.
The convolution equation $Y_m * Y_n = Y_{m+n},\ m, n \in \mathbb{C}$, holds. The *fractional differential operator* is defined as $D^m := I^{-m} := Y_{-m}*$. If arbitrary $m \in \mathbb{C}$ are admitted, every fractional integral operator $I^m = D^{-m}$ can be interpreted as a differential one, and vice versa. The equation

$$\begin{aligned} &\forall m \in \mathbb{C} \setminus (-\mathbb{N}) \forall \epsilon > 0 \forall \varphi \in \mathrm{C}_0^\infty \text{ with } \varphi|_{[0,\epsilon]} = 0: \\ &Y_m(\varphi) = \Gamma(m)^{-1} \int_{\epsilon}^{\infty} t^{m-1} \varphi(t) dt \end{aligned} \tag{1.68}$$

suggested to call, for $m \in \mathbb{C}$ with $\Re(m) \leq 0$, the distribution Y_m the *finite part* of the function $\left(\Gamma(m)^{-1} t^{m-1}\right)_{\mathbb{R}}$ [67, (II,2;26)].
The distribution $Y_m,\ m \in \mathbb{C}$, is Laplace transformable, and indeed

$$\begin{aligned} &\mathcal{L}(Y_m)(s) = s^{-m} := e^{-m \ln(s)},\ \Re(s) > 0,\ \text{where} \\ &s = |s| e^{j\alpha},\ |s| > 0,\ -\pi/2 < \alpha < \pi/2,\ \ln(s) = \ln(|s|) + j\alpha,\ j = \sqrt{-1}. \end{aligned} \tag{1.69}$$

Let $\mu > 0$ be a fixed positive real number. Then one usually calls I^μ an integral operator and $D^\mu = I^{-\mu}$ a differential operator. The convolution equations $Y_{m\mu} * Y_{n\mu} = Y_{(m+n)\mu}$ imply that

$$\bigoplus_{m \in \mathbb{Z}} \mathbb{C} Y_{m\mu} \subset \left(\mathrm{C}_+^{-\infty}, *\right),\ Y_{\mu m} = Y_\mu^m,\ Y_\mu^{-1} = Y_{-\mu}, \tag{1.70}$$

is a subalgebra of $\mathrm{C}_+^{-\infty}$ with respect to the convolution multiplication. Let $\mathbb{C}[s, s^{-1}] = \bigoplus_{m \in \mathbb{Z}} \mathbb{C} s^m$ with $s^m s^n = s^{m+n}$ denote the principal ideal domain of *Laurent polynomials*. Then the map

$$\mathbb{C}[s, s^{-1}] \to \bigoplus_{m \in \mathbb{Z}} \mathbb{C} Y_{m\mu},\ H = \sum_{m \in \mathbb{Z}} a_m s^m \mapsto H(Y_\mu) := \sum_{m \in \mathbb{Z}} a_m Y_\mu^m,\ Y_\mu^m = Y_{m\mu}, \tag{1.71}$$

is an algebra isomorphism. Hence $\mathrm{C}_+^{-\infty}$ is a $\mathbb{C}[s,s^{-1}]$-module with the scalar multiplication

$$\begin{aligned} &H \circ_\mu y = H(Y_\mu) * y,\ H \in \mathbb{C}[s,s^{-1}],\ y \in \mathrm{C}_+^{-\infty} \\ &\Longrightarrow H(Y_\mu) * y = \sum_{m\in\mathbb{Z}} a_m \left(I^\mu\right)^m y = \sum_{m\in\mathbb{Z}} a_{-m}(D^\mu)^m y. \end{aligned} \tag{1.72}$$

An equation $H \circ_\mu y = u$ with given $H \in \mathbb{C}[s,s^{-1}]$ and $u \in \mathrm{C}_+^{-\infty}$ is called an *inhomogeneous (μ)-fractional integral/differential equation* or, shorter, fractional differential equation.
We are now going to extend these equations considerably. Let $\mathbb{C}\langle\langle s\rangle\rangle$ denote the field of convergent Laurent series at 0 with at most a pole at 0. This is given by

$$\mathbb{C}\langle\langle s\rangle\rangle = \mathbb{C}[s^{-1}] \bigoplus \mathbb{C}\langle s\rangle_+,\ \mathbb{C}\langle s\rangle_+ := \left\{ \sum_{m=1}^\infty a_m s^m;\ a_m \in \mathbb{C},\ \limsup_m \sqrt[m]{|a_m|} < \infty \right\}. \tag{1.73}$$

The series $\sum_{m=1}^\infty a_m s^m$ is a (locally at 0) convergent power series with constant term 0 and the convergence radius $\rho := \left(\limsup_m \sqrt[m]{|a_m|}\right)^{-1} > 0$, hence holomorphic in the disc $\{s \in \mathbb{C};\ |s| < \rho\}$. We write

$$H = H_- + H_+,\ H_- := \sum_{m=0}^\infty a_{-m} s^{-m} \in \mathbb{C}[s^{-1}],\ H_+ := \sum_{m=1}^\infty a_m s^m \in \mathbb{C}\langle s\rangle_+. \tag{1.74}$$

Almost all a_{-m} for $m \in \mathbb{N}$ are 0. We are now going to define $H(Y_\mu) := H_-(Y_\mu) + H_+(Y_\mu)$ where, of course, $H_-(Y_\mu) = \sum_{m=0}^\infty a_{-m} Y_{-m\mu}$. The infinite sum $H_+(Y_\mu) := \sum_{m=1}^\infty a_m Y_{m\mu}$ is not defined a priori in the distribution space $\mathrm{C}_+^{-\infty}$ and hence we proceed as follows. We define

$$\begin{aligned} &\widehat{H_+}_\mu(z) := \sum_{m=0}^\infty a_{m+1}\Gamma((m+1)\mu)^{-1} z^m,\ z \in \mathbb{C}, \\ &H_+(Y_\mu) := \left(t^{\mu-1}\widehat{H_+}_\mu(t^\mu)\right)_{\mathbb{R}},\ H(Y_\mu) := H_-(Y_\mu) + H_+(Y_\mu), \end{aligned} \tag{1.75}$$

where $\widehat{H_+}_\mu(z)$ is an everywhere convergent power series and an entire holomorphic function on $\mathbb{C}$. This holds since the $\Gamma((m+1)\mu)$ grow very fast like factorials, due to Stirling's formula for the Γ-function. Hence $\widehat{H_+}_\mu(t^\mu)$ is continuous on $[0,\infty)$ and $H_+(Y_\mu) := \left(t^{\mu-1}\widehat{H_+}_\mu(t^\mu)\right)_{\mathbb{R}}$ is locally integrable on $\mathbb{R}$ since $\mu - 1 > -1$. In particular, $H_+(Y_\mu) * u,\ u \in \mathrm{C}_+^{0,\mathrm{pc}}$, is continuous. We show

$$\begin{aligned} &\sum_{m\in\mathbb{Z}} a_m Y_{m\mu} := \sum_{m=1}^\infty a_{-m} Y_{-m\mu} + \lim_{N\to\infty} \sum_{m=1}^N a_m Y_{m\mu} = H(Y_\mu),\ \text{i.e.,} \\ &\quad \forall \varphi \in \mathrm{C}_0^\infty : \sum_{m\in\mathbb{Z}} a_m Y_{m\mu}(\varphi) \\ &\qquad := \sum_{m=1}^\infty a_{-m} Y_{-m\mu}(\varphi) + \lim_{N\to\infty} \sum_{m=1}^N a_m Y_{m\mu}(\varphi) = H(Y_\mu)(\varphi) \end{aligned} \tag{1.76}$$

and

$$\forall H_1, H_2 \in \mathbb{C}\langle\langle s\rangle\rangle : \ (H_1 + / \cdot H_2)(Y_\mu) = H_1(Y_\mu) + / * H_2(Y_\mu). \tag{1.77}$$

Therefore the map

$$\mathbb{C}\langle\langle s\rangle\rangle \cong \mathcal{F}_{2,\mu} := \{H(Y_\mu);\ H \in \mathbb{C}\langle\langle s\rangle\rangle\}\,,\ H \mapsto H(Y_\mu), \tag{1.78}$$

is a field isomorphism where $\mathcal{F}_{2,\mu}$ is a large subfield of $\left(\mathrm{C}_+^{-\infty}, *\right)$. This, in turn, implies that $\mathrm{C}_+^{-\infty}$ is a $\mathbb{C}\langle\langle s\rangle\rangle$-vector space with the scalar multiplication

$$H \circ_\mu u = H(Y_\mu) * u,\ H \in \mathbb{C}\langle\langle s\rangle\rangle,\ u \in \mathrm{C}_+^{-\infty},\ H \circ_\mu \delta = H(Y_\mu). \tag{1.79}$$

Then $H \circ_\mu \delta = H(Y_\mu)$ is called the *μ-impulse response* of H. As usual the action $\circ_\mu$ is extended to matrices and vectors. In particular, we consider linear systems

$$\begin{aligned}
&P \circ_\mu y = P(Y_\mu) * y = Q \circ_\mu u = Q(Y_\mu) * u \text{ where } u \in \left(\mathrm{C}_+^{-\infty}\right)^m,\ y \in \left(\mathrm{C}_+^{-\infty}\right)^p,\\
&(P, -Q) \in \mathbb{C}\langle\langle s\rangle\rangle^{p\times(p+m)},\ \mathrm{rank}(P) = p \text{ or } \det(P) \neq 0,\ H := P^{-1}Q\\
&\Longrightarrow y = H \circ_\mu u = H(Y_\mu) * u \Longrightarrow\\
&\left(\mathrm{C}_+^{-\infty}\right)^m \underset{\mathbb{C}\langle\langle s\rangle\rangle}{\cong} \mathcal{B} := \left\{ \left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right) \in \left(\mathrm{C}_+^{-\infty}\right)^{p+m} \quad P \circ_\mu y = Q \circ_\mu u \right\},\ u \mapsto \left(\begin{smallmatrix} H\circ_\mu u \\ u \end{smallmatrix}\right).
\end{aligned} \tag{1.80}$$

The solution $\mathbb{C}\langle\langle s\rangle\rangle$-vector space $\mathcal{B}$ is called a *generalized fractional IO behavior*. By definition its trajectories have left bounded support like the signals in connection with the Laplace transform in this book and like the often used signals in electrical engineering. Initial conditions are neither needed nor used in our approach. If, in particular, the input u is of the general form $u = H_2 \circ_\mu \delta = H_2(Y_\mu)$ where $H_2 \in \mathbb{C}\langle\langle s\rangle\rangle^m$ then

$$y := HH_2 \circ_\mu \delta = (HH_2)(Y_\mu) \text{ solves } P(Y_\mu) * y = Q(Y_\mu) * H_2(Y_\mu) \tag{1.81}$$

uniquely and $y = (HH_2)(Y_\mu)$ can be explicitly computed. Standard multivariable μ-fractional integral/differential systems are the special case where $(P, -Q) \in \mathbb{C}[s, s^{-1}]^{p\times(p+m)}$. For instance, consider the binomial power series

$$\begin{aligned}
H :&= (1-\lambda s)^{-k} = \sum_{m=0}^{\infty} \left(\begin{smallmatrix} -k \\ m \end{smallmatrix}\right) \lambda^m s^m,\ 0 \neq \lambda \in \mathbb{C},\ k \geq 1,\\
\Longrightarrow H &= \sum_{m=0}^{\infty} \left(\begin{smallmatrix} m+k-1 \\ k-1 \end{smallmatrix}\right) \lambda^m s^m = 1 + H_+,\\
\widehat{H_+}_\mu(z) &= \sum_{m=0}^{\infty} \left(\begin{smallmatrix} m+k \\ k-1 \end{smallmatrix}\right) \lambda^{m+1} \Gamma((m+1)\mu)^{-1} z^m,\\
(1-\lambda s)^{-k}(Y_\mu) &= (\delta - \lambda Y_\mu)^{-k} = \delta + {}_\lambda Z_\mu^{(k)},\ {}_\lambda Z_\mu^{(k)} := \left(t^{\mu-1}\widehat{H_+}_\mu(t^\mu)\right)_{\mathbb{R}}.
\end{aligned} \tag{1.82}$$

Note that for $0 < \mu < 1$ the function ${}_\lambda Z_\mu^{(k)}$ is locally integrable, but not piecewise continuous at 0.

Let, more generally, $H \in \mathbb{C}(s)$ be an arbitrary rational function with its partial fraction decomposition, cf. Section 4.5.2,

$$\begin{aligned} H &= \sum_{m\in\mathbb{Z}} a_m s^m + \sum_{0\neq\lambda\in\mathrm{pole}(H)} \sum_{k=1}^{m_\lambda} a_{\lambda,k}(s-\lambda)^{-k} \\ &= \sum_{m\in\mathbb{Z}} a_m s^m + \sum_{0\neq\lambda\in\mathrm{pole}(H)} \sum_{k=1}^{m_\lambda} a_{\lambda,k}(-\lambda)^{-k}(1-\lambda^{-1}s)^{-k} \text{ with} \\ & a_m, a_{\lambda,k} \in \mathbb{C},\ 1 \le m_\lambda \in \mathbb{N},\ a_{\lambda,m_\lambda} \neq 0,\ a_m = 0 \text{ for almost all } m. \end{aligned} \tag{1.83}$$

Then

$$\begin{aligned} H(Y_\mu) &= \sum_{m\in\mathbb{Z}} a_m Y_{m\mu} + \sum_{0\neq\lambda\in\mathrm{pole}(H)} \sum_{k=1}^{m_\lambda} a_{\lambda,k}(-\lambda)^{-k}(1-\lambda^{-1}s)^{-k}(Y_\mu) \\ &= \sum_{m\in\mathbb{Z}} a_m Y_{m\mu} + \sum_{0\neq\lambda\in\mathrm{pole}(H)} \sum_{k=1}^{m_\lambda} a_{\lambda,k}(-\lambda)^{-k}\left(\delta + {}_{\lambda^{-1}}Z_\mu^{(k)}\right). \end{aligned} \tag{1.84}$$

For $H \in \mathbb{C}\langle\langle s\rangle\rangle$ the map $H(Y_\mu)* = H\circ_\mu$ induces a map $H(Y_\mu)*:\ \mathrm{C}_+^{0,\mathrm{pc}} \to \mathrm{C}_+^{0,\mathrm{pc}}$ if and only if 0 is not a pole of H, i.e. if H is a locally convergent power series or $a_m = 0$ for $m < 0$. The distribution $H(Y_\mu)$ is even a locally integrable function on $\mathbb{R}$ if and only if in addition $H(0) = a_0 = 0$.
For a rational function $H \in \mathbb{C}(s)$ the Laplace transform of $H(Y_\mu)$ is

$$\mathcal{L}(H(Y_\mu)) = H(s^{-\mu}),\ H \in \mathbb{C}(s). \tag{1.85}$$

In our approach to fractional differential equations this result is not applied. We do not know an analogue of (1.85) for general convergent Laurent series.
In practice only rational exponents $\mu \in \mathbb{Q}$ are considered. Assume a positive rational number

$$\begin{aligned} &\mu = r/m,\ r,m \in \mathbb{N},\ r,m > 0 \\ &\Longrightarrow H(Y_\mu) = H(s^r)(Y_{1/m}),\ H(s)\circ_\mu u = H(s^r)\circ_{1/m} u. \end{aligned} \tag{1.86}$$

For finitely many positive rational numbers μ_i with their least common denominator $m \in \mathbb{N}$ this implies

$$\mu_i = r_i/m,\ \ r_i > 0,\ H(Y_{\mu_i}) = H(s^{r_i})(Y_{1/m}),\ H(s)\circ_{\mu_i} u = H(s^{r_i})\circ_{1/m} u. \tag{1.87}$$

Hence finitely many different operators $I^{\mu_i} = Y_{\mu_i}*$ with positive rational indices μ_i and their differential counterparts $D^{\mu_i} = Y_{-\mu_i}*$ can be treated with the single vector space $\left({}_{\mathbb{C}\langle\langle s\rangle\rangle}\mathrm{C}_+^{-\infty}, \circ_{1/m}\right)$. Note that

$$\begin{aligned} &s\circ\delta = s^{-1}\circ_1\delta = \delta' = Y_{-1} \\ &\Longrightarrow \forall H \in \mathbb{C}(s):\ H(s)\circ\delta = H(s^{-1})\circ_1\delta = H(s^{-1})(Y_1),\ Y_1 = Y. \end{aligned} \tag{1.88}$$

For a nonrational convergent Laurent series $H(s)$ the function $H(s^{-1})$ does not belong to $\mathbb{C}\langle\langle s\rangle\rangle$ and $H(s^{-1})(Y_1)$ is not defined.
The preceding theory can be reformulated as a theory for the vector space $\left({}_{\mathbb{C}\langle\langle s^{1/\infty}\rangle\rangle}\mathrm{C}_+^{-\infty}, \circ_1\right)$ where

$$\mathbb{C}\langle\langle s^{1/\infty}\rangle\rangle := \bigcup_{m=1}^{\infty} \mathbb{C}\langle\langle s^{1/m}\rangle\rangle \tag{1.89}$$

is the *Puiseux field* of convergent *Puiseux series* that is the algebraic closure of the field $\mathbb{C}\langle\langle s\rangle\rangle$ of convergent Laurent series, cf. [Wikipedia; https://en.wikipedia.org/wiki/Puiseux series, 8 September 8 2019], [15, §3.1] and Chapter 12, and where the scalar multiplication is given by

$$\begin{aligned}&H(s^{1/m})\circ_1 u := H(s)\circ_{1/m} u \text{ where}\\&H\in\mathbb{C}\langle\langle s\rangle\rangle,\ H(s^{1/m})\in\mathbb{C}\langle\langle s^{1/m}\rangle\rangle\subset\mathbb{C}\langle\langle s^{1/\infty}\rangle\rangle,\ u\in \mathrm{C}_+^{-\infty}.\end{aligned}\tag{1.90}$$

A different application of $\mathbb{C}\langle\langle s^{1/\infty}\rangle\rangle$ for LTV-(linear time-varying) systems was described in [15].
Finally we treat a simple example that already demonstrates the power of our method for the explicit solution of fractional differential equations. Consider

$$\begin{aligned}&(D^{1/2}-\lambda_1)y = e^{\lambda_2 t}Y,\ \lambda_1\neq 0,\ \lambda_2\neq 0,\ \lambda_3^2=\lambda_2\neq\lambda_1^2,\ \mu:=1/2,\ \text{or}\\&(s^{-1}-\lambda_1)\circ_{1/2} y = (s-\lambda_2)^{-1}\circ\delta = (s^{-1}-\lambda_2)^{-1}\circ_1\delta = (s^{-2}-\lambda_2)^{-1}\circ_{1/2}\delta.\end{aligned}\tag{1.91}$$

This fractional differential equation has the unique solution

$$\begin{aligned}y &= (s^{-1}-\lambda_1)^{-1}(s^{-2}-\lambda_2)^{-1}\circ_{1/2}\delta\\&= \left(s^3(1-\lambda_1 s)^{-1}(1-\lambda_3 s)^{-1}(1+\lambda_3 s)^{-1}\right)(Y_{1/2}).\end{aligned}\tag{1.92}$$

The partial fraction decomposition is

$$\begin{aligned}&s^3(1-\lambda_1 s)^{-1}(1-\lambda_3 s)^{-1}(1+\lambda_3 s)^{-1}\\&\qquad = a+b(1-\lambda_1 s)^{-1}+c(1-\lambda_3 s)^{-1}+d(1+\lambda_3 s)^{-1}\ \text{with}\\&a=(\lambda_1\lambda_2)^{-1},\ b=\left(\lambda_1(\lambda_1^2-\lambda_2)\right)^{-1},\ c=(2\lambda_2(\lambda_3-\lambda_1))^{-1},\\&d=-\left(2\lambda_2(\lambda_3+\lambda_1)\right)^{-1}\in\mathbb{C},\ a+b+c+d\underset{s=0}{=}0,\end{aligned}$$

and furnishes the continuous solution, cf. (1.84),

$$\begin{aligned}y &= a\delta + b\left(\delta+{}_{\lambda_1}Z_{1/2}^{(1)}\right)+c\left(\delta+{}_{\lambda_3}Z_{1/2}^{(1)}\right)+d\left(\delta+{}_{(-\lambda_3)}Z_{1/2}^{(1)}\right)\\&= b\,{}_{\lambda_1}Z_{1/2}^{(1)}+c\,{}_{\lambda_3}Z_{1/2}^{(1)}+d\,{}_{(-\lambda_3)}Z_{1/2}^{(1)}\end{aligned}\tag{1.93}$$

with the locally integrable functions ${}_{\lambda}Z_{\mu}^{(k)}$ on $\mathbb{R}$.

1.6 Stability

A basic requirement for an IO behavior $\mathcal{B}$ from (1.11) with transfer matrix H is its *stability*. We study this by means of the *Chinese Remainder Theorem* (CRT) in Chapter 4, cf. [52]. An important consequence of the latter is the *primary direct sum decomposition* of the torsion submodule $\mathrm{t}(M)$ of any $\mathbb{C}[s]$-module M, cf. Theorem 4.3.2, viz.

$$\begin{aligned}\mathrm{t}(M) &:= \{y\in M;\ \exists 0\neq f\in\mathbb{C}[s]:\ f\circ y=0\}\\&= \bigoplus_{\lambda\in\mathbb{C}} M_\lambda \ni y = \sum_\lambda y_\lambda,\ M_\lambda := \left\{y\in M;\ \exists k\in\mathbb{N}\text{ with }(s-\lambda)^k\circ y=0\right\}.\end{aligned}\tag{1.94}$$

The CRT yields the y_λ from y constructively. In particular, one gets

$$\mathrm{t}\left(\mathrm{C}^{-\infty}\right)_\lambda = \mathrm{t}\left(\mathrm{C}^{\infty}\right)_\lambda = \mathbb{C}[t]e^{\lambda t} \quad \text{(Section 4.3.3)}. \tag{1.95}$$

The autonomous part $\mathcal{B}^0$ of $\mathcal{B}$ thus admits the primary decomposition $\mathcal{B}^0 = \bigoplus_{\lambda\in\mathbb{C}} \mathcal{B}^0_\lambda$. If $\mathcal{B}^0_\lambda$ is nonzero, the number λ is called a *characteristic value* or *pole* of $\mathcal{B}^0$ and of $\mathcal{B}$. Since $\dim_{\mathbb{C}}(\mathcal{B}^0) < \infty$, this occurs only for finitely many λ, and indeed, cf. Section 4.4.1,

$$\begin{aligned} \mathrm{char}(\mathcal{B}^0) &:= \left\{\lambda \in \mathbb{C};\ \mathcal{B}^0_\lambda \neq 0\right\} = \mathrm{V}_{\mathbb{C}}(\det(P)) \\ &= \{\lambda \in \mathbb{C};\ \mathrm{rank}(P(\lambda)) < p = \mathrm{rank}(P)\} \\ \Longrightarrow \mathcal{B}^0 &= \bigoplus_{\lambda\in\mathrm{char}(\mathcal{B}^0)} \mathcal{B}^0_\lambda,\ \mathcal{B}^0_\lambda = \mathcal{B}^0 \bigcap \mathbb{C}[t]^p e^{\lambda t} \subset \mathrm{t}(\mathcal{F})^p. \end{aligned} \tag{1.96}$$

The set $\mathrm{char}(\mathcal{B}^0)$ is called the *characteristic variety* of $\mathcal{B}^0$, a term originally from *Algebraic Analysis* [26], [58], [8]. The trajectories in $\mathcal{B}^0_\lambda$, $\lambda \in \mathrm{char}(\mathcal{B}^0)$, are called the *$\lambda$-modes of $\mathcal{B}^0$*. The finite dimension $\mathrm{l}_\lambda(\mathcal{B}^0) := \mathrm{mult}(\mathcal{B}^0_\lambda) := \dim_{\mathbb{C}}\left(\mathcal{B}^0_\lambda\right)$ is called the *λ-length or λ-multiplicity* of $\mathcal{B}^0$ and gives rise to the infinite vector

$$\begin{aligned} &\mathrm{l}(\mathcal{B}^0) = \left(\mathrm{l}_\lambda(\mathcal{B}^0)\right)_{\lambda\in\mathbb{C}} \in \mathbb{N}^{(\mathbb{C})} \\ &:= \left\{\mu = (\mu(\lambda))_{\lambda\in\mathbb{C}} \in \mathbb{N}^{\mathbb{C}};\ \mathrm{supp}(\mu) := \{\lambda \in \mathbb{C};\ \mu(\lambda) \neq 0\} \text{ finite}\right\}. \end{aligned} \tag{1.97}$$

In Theorem 4.4.11 we compute a $\mathbb{C}$-basis of the space $\mathcal{B}^0_\lambda$of λ-modes, and hence the length $\mathrm{l}_\lambda(\mathcal{B}^0)$ and a basis of $\mathcal{B}^0$. In the literature [13, §7.2, §13.4.2], [72, §2.5] the elements $\mu \in \mathbb{N}^{(\mathbb{C})}$ are often written as

$$\mathrm{supp}(\mu) = \{\lambda_1, \cdots, \lambda_r\},\ \mu = \left\{\underbrace{\lambda_1, \cdots, \lambda_1}_{\mu(\lambda_1)}, \cdots, \underbrace{\lambda_r, \cdots, \lambda_r}_{\mu(\lambda_r)}\right\}. \tag{1.98}$$

The set $\mathrm{supp}(\mu)$ with the multiplicities $\mu(\lambda_i)$ is called a finite *valued subset of* $\mathbb{C}$. Notice that these μ can be added in $\mathbb{N}^{(\mathbb{C})}$ and subtracted in $\mathbb{Z}^{(\mathbb{C})}$, as is done in [72, Thm. 2.62] without detailed explanation. One writes $\mu \uplus \nu := \mu + \nu$. The multiplicities play an important part in connection with poles of *Rosenbrock equations*, cf. Section 5.3.5.
The state space representation (1.28) implies

$$\begin{aligned} &\mathcal{B}^0_s = \left\{x = e^{tA}x(0); x(0) \in \mathbb{C}^n\right\} \cong \mathcal{B}^0 \text{ and} \\ &\mathrm{char}(\mathcal{B}^0) = \mathrm{char}(\mathcal{B}^0_s) = \mathrm{V}_{\mathbb{C}}(\det(s\,\mathrm{id}_n - A)) = \mathrm{spec}(A), \end{aligned} \tag{1.99}$$

where $\mathrm{spec}(A)$ is the *spectrum* or *set of eigenvalues* of A. The primary components $(\mathcal{B}^0_s)_\lambda$ are related to the Jordan decomposition of A. This, in turn, is given by the *primary or Jordan decomposition* of $\mathbb{C}^n = \bigoplus_{\lambda\in\mathrm{spec}(A)}(\mathbb{C}^n)_\lambda$ into the *generalized eigenspaces* $(\mathbb{C}^n)_\lambda$ of A where $\mathbb{C}^n$ is a $\mathbb{C}[s]$ -module via $s \circ x = Ax$, cf. Section 4.5.1.
In contrast to $\mathrm{char}(\mathcal{B}^0)$ the *characteristic variety of $\mathcal{B}$* is

$$\begin{aligned} &\mathrm{char}(\mathcal{B}) := \{\lambda \in \mathbb{C};\ \mathrm{rank}(P(\lambda), -Q(\lambda)) < p = \mathrm{rank}(P, -Q)\} \\ &\Longrightarrow \mathrm{char}(\mathcal{B}^0) = \mathrm{char}(\mathcal{B}) \cup \mathrm{pole}(H),\ \text{cf. Theorems 5.2.7, 5.2.9.} \end{aligned} \tag{1.100}$$

The behavior $\mathcal{B}$ is controllable, i.e., its module is free, if and only if $\mathrm{char}(\mathcal{B}) = \emptyset$. Therefore the elements of $\mathrm{pole}(H)$ resp. of $\mathrm{char}(\mathcal{B})$ are called the *controllable* resp. *uncontrollable* poles of $\mathcal{B}$. Note that $\mathrm{pole}(H) \cap \mathrm{char}(\mathcal{B}) \neq \emptyset$ may occur.
The $\mathbb{C}[s]$-module of *asymptotically stable polynomial-exponential signals* is

$$\mathcal{F}_- := \left\{ y \in \mathrm{t}(\mathcal{F});\ \lim_{t\to\infty} y(t) = 0 \right\} = \bigoplus_{\lambda \in \mathbb{C}_-} \mathbb{C}[t]e^{\lambda t},\ \mathbb{C}_- := \{\lambda \in \mathbb{C};\ \Re(\lambda) < 0\}, \tag{1.101}$$

cf. Theorem 4.4.16. The behaviors $\mathcal{B}$ and $\mathcal{B}^0$ are called *asymptotically stable if and only if*

$$\forall z \in \mathcal{B}^0 :\ \lim_{t\to\infty} z(t) = 0 \Longleftrightarrow \mathcal{B}^0 \subset \mathcal{F}_-^p \Longleftrightarrow \mathrm{char}(\mathcal{B}^0) \subset \mathbb{C}_-. \tag{1.102}$$

Recall the decompositions $y = y_{ss} + z,\ z \in \mathcal{B}^0$, into the *steady or stationary state* y_{ss} and the *transient* z from (1.26), (1.43) and (1.59). If $\mathcal{B}^0$ is asymptotically stable and hence $\lim_{t\to\infty}(y - y_{ss})(t) = 0$, y and y_{ss} can often be identified in practical situations. This suggested the *steady (stationary) state, transient* terminology, that is also used, but not justified without the asymptotic stability of $\mathcal{B}^0$. It would be appropriate to talk of *one* instead of *the* steady state y_{ss} of y, but all satisfy $\lim_{t\to\infty}(y - y_{ss})(t) = 0$.
The *external stability* of $\mathcal{B}$ is a property of its transfer operator $H\circ$. We assume that H is proper, and obtain the operator $H\circ : \left(\mathrm{C}_+^{0,\mathrm{pc}}\right)^m \to \left(\mathrm{C}_+^{0,\mathrm{pc}}\right)^p$. We need the normed signal spaces L^q and L_+^q, $1 \le q \le \infty$, defined by

$$\begin{aligned} \mathrm{L}^q &:= \left\{ u \in \mathrm{C}^{0,\mathrm{pc}};\ \|u\|_q := \left(\int_{-\infty}^{\infty} |u(t)|^q dt \right)^{1/q} < \infty \right\},\ q < \infty, \\ \mathrm{L}^\infty &:= \left\{ u \in \mathrm{C}^{0,\mathrm{pc}};\ \|u\|_\infty := \sup_{t\in\mathbb{R}} |u(t)| < \infty \right\},\ \mathrm{L}_+^q := \mathrm{L}^q \cap \mathrm{C}_+^{0,\mathrm{pc}},\ q \le \infty, \end{aligned} \tag{1.103}$$

with the norms $\| - \|_q$. The completions $\mathfrak{L}^q$ of L^q for $q = 1, 2, \infty$ are Banach spaces and used in Section 9.2.4 in connection with the *robustness of stabilizing compensators*. As usual, we also consider matrices with entries in these L_+^q. Let

$$\begin{aligned} &H = H_0 + H_{\mathrm{spr}},\ H_0 \in \mathbb{C}^{p\times m},\ h_{\mathrm{spr}} := H_{\mathrm{spr}} \circ \delta \in \left(\mathrm{C}_+^{0,\mathrm{pc}}\right)^{p\times m} \\ &\Longrightarrow \forall u \in \left(\mathrm{C}_+^{0,\mathrm{pc}}\right)^m :\ H \circ u = H_0 u + H_{\mathrm{spr}} \circ u = H_0 u + h_{\mathrm{spr}} * u. \end{aligned} \tag{1.104}$$

External stability of the behavior is then characterized by the following equivalent properties, cf. Theorem 7.2.83, Corollary 7.2.84:

$$\begin{aligned} &(i) \quad \mathrm{pole}(H) \subset \mathbb{C}_-, \\ &(ii) \quad h_{\mathrm{spr}} \in \left(\mathrm{L}_+^1\right)^{p\times m}, \\ &(iii) \quad \text{for } q = 1 \text{ or } q = \infty :\ H \circ \left(\mathrm{L}_+^q\right)^m \subseteq \left(\mathrm{L}_+^q\right)^p, \\ &(iv) \quad \forall q,\ 1 \le q \le \infty :\ H \circ \left(\mathrm{L}_+^q\right)^m \subseteq \left(\mathrm{L}_+^q\right)^p. \end{aligned} \tag{1.105}$$

Moreover the operator $H\circ$ is continuous in the $\|-\|_q$-norms, i.e., the output $H\circ u$ depends continuously on the input u. The condition (iii) for $q = \infty$ is called *BIBO (bounded input/bounded output) stability*. Since $\mathrm{pole}(H) \subseteq \mathrm{char}(\mathcal{B}^0)$ we infer that asymptotic stability implies external stability.

1.7 Electrical and mechanical networks

The theory of *electrical networks* is both a very important source and application of systems theoretic methods, and is the only applied field that is discussed in detail in this book, cf. Sections 2.1.3 and 7.3. According to [70],[2], [66] the following methods and results are fundamental or even the most basic results of electrical engineering. We derive them with the systems theory of this book that is very suitable for exact mathematical derivations in this field. Several of our equations are more general than those of the cited books. Mechanical networks are then treated via the *electrical-mechanical Firestone analogy*, cf. Section 7.4. Examples 7.3.17, 7.3.25, 7.3.34 and 7.4.6 show how the theorems and algorithms are applied. We discuss *translational mechanical networks*, but not *rotational* ones. We refer to the books [4], [38], [41] on *mechatronics* where networks of additional energy domains and their interconnections are discussed. The mathematics of the present book is also useful for these extensions. We do not discuss the vast design part of electrical and mechanical engineering, i.e., the construction of an electrical network and not just of an arbitrary IO behavior with prescribed transfer matrix. For the latter Kalman's realization theorem solves the problem, cf. (1.28) and Chapter 11.
In electrical and mechanical engineering IO behaviors, i.e., with a decomposition of the trajectories into input and output components, are far more important than general behaviors, for instance for steady state and superposition principle considerations. This is in contrast to Willems' general philosophy.
Electrical networks give rise to behaviors of a special form. We use the real base field $\mathbb{R}$ and the real versions $\mathcal{F} := \mathcal{F}_{\mathbb{R}}$ of the injective cogenerator function modules, for instance $\mathcal{F}_{\mathbb{R}} := \mathrm{C}^{\infty}(\mathbb{R},\mathbb{R})$ or $\mathcal{F}_{\mathbb{R}} := \mathrm{C}^{-\infty}(\mathbb{R},\mathbb{R})$, that are later precisely explained. The notion of a *network* refers to a *connected directed graph* (V,K), consisting of a finite set V of size $m := \sharp(V)$ of *nodes* or *vertices* and a finite set K of size $n := \sharp(K)$ of *branches, edges* or *arrows* with two maps $\mathrm{dom}, \mathrm{cod} : K \to V$ (domain, codomain), written as $k : v := \mathrm{dom}(k) \to w := \mathrm{cod}(k)$. Then k is called a *directed* branch from the node v to the node w. The connectedness means that for arbitrary $v, w \in V$ there is a path along edges from v to w. In a real electrical network a branch $k : v \to w$ is realized by a wire (short circuit), a voltage or current source or a passive electrical element with two terminals. The nodes represent the points where the wires or terminals of the different electrical elements are connected. The trajectories of the network behavior are of the form

$$\left(\begin{smallmatrix} U \\ I \end{smallmatrix}\right) \in \mathcal{F}^{K \uplus K},\ U := (U_k)_{k \in K} \in \mathcal{F}^K,\ I := (I_k)_{k \in K} \in \mathcal{F}^K, \tag{1.106}$$

where U_k is the *voltage or potential difference* between v and w and I_k is the *current* through k from v to w. The set K is decomposed as $K = K_p \uplus K_s$ where K_p resp. K_s contain the *one-port resp. the source branches.* Along $k \in K_p$ there is an electrical device with *two terminals*, called 2-*pole or one-port*, described by an equation

$$P_k \circ U_k = Q_k \circ I_k,\ P_k, Q_k \in \mathbb{R}[s],\ P_k \neq 0,\ Q_k \neq 0. \tag{1.107}$$

The prototypical one-port branches are the ideal *resistance, capacitance, inductance* (R, C, L)-branches with the simple equations

$$U_k = R_k I_k,\ C_k s \circ U_k = I_k,\ U_k = L_k s \circ I_k,\ R_k, C_k, L_k > 0. \tag{1.108}$$

Voltage resp. current source branches $k \in K_s$ are characterized by given U_k resp. I_k that are supplied to the network from outside, whereas the corresponding I_k resp. U_k are determined by the network. The network without the voltage or current sources $U_k, I_k,\ k \in K_s$, is called *passive* and often studied. However, we always include the sources into the considerations.
The U_k of the network satisfy *Kirchhoff's circuit* or *voltage law* (KVL) and the currents I_k *Kirchhoff's node* or *current law* (KCL) that will be specified in Section 2.1.3. With these data the behavior of the network is

$$\mathcal{B} := \left\{ \left(\begin{smallmatrix} U \\ I \end{smallmatrix}\right) \in \mathcal{F}_{\mathbb{R}}^{K \uplus K};\ \begin{cases} (i)(\text{KVL}) \text{ and } (\text{KCL}) \text{ are satisfied} \\ (ii)\ \forall k \in K_p:\ P_k \circ U_k = Q_k \circ I_k \end{cases} \right\}. \tag{1.109}$$

There are networks with different equations (ii), for instance with *ideal transformers* or *gyrators* or *controlled voltage* or *current sources*. These do not change the mathematics essentially, cf. Corollaries 7.3.14 and 7.3.22. Let

$$V_s := \{\mathrm{dom}(k), \mathrm{cod}(k);\ k \in K_s\}\,,\ m_s := \sharp(V_s),\ n_s := \sharp(K_s),\ m_s \le 2n_s. \tag{1.110}$$

The nodes in V_s are called the *terminals or poles* of $\mathcal{B}$, and represent the connection with the outside, and $\mathcal{B}$ is called an m_s*-pole*. If $m_s = 2n_s$, i.e., if the $\mathrm{dom}(k), \mathrm{cod}(k),\ k \in K_s$, are pairwise distinct, $\mathcal{B}$ is called an n_s*-port*, and each $k : \mathrm{dom}(k) \to \mathrm{cod}(k),\ k \in K_s$, is called a *port*. New source branches between existing nodes can be added to K_s, but change the network and its behavior. In the engineering literature more special graphs are usually employed.
Usually the study of $\mathcal{B}$ begins with the *node-potential*, *mesh-current* or *state space* method, based on graph theory, to derive the consequences of the Kirchhoff laws, cf. [70, §3.1-4], [2, Ch. 3]. These methods are, however, only special cases of the *Gauß algorithm* for the solution of linear systems over a field, and we can and do therefore proceed with a much simpler method. Indeed, let $A = (A_{vk})_{v \in V, k \in K} \in \mathbb{R}^{V \times K}$ denote the *incidence matrix* of (V, K), defined by

$$A_{vk} = A(v,k) = \begin{cases} 1 & \text{if } v = \mathrm{dom}(k) \ne \mathrm{cod}(k) \\ -1 & \text{if } v = \mathrm{cod}(k) \ne \mathrm{dom}(k) \\ 0 & \text{otherwise} \end{cases}. \tag{1.111}$$

The connectedness of (V, K) implies $\mathrm{rank}(A) = m - 1,\ m = \sharp(V)$. Elementary row operations and column permutations on A furnish the echelon form

$$\begin{aligned} &XA = \begin{array}{c c} & \begin{array}{cc} K_1 & K_2 \end{array} \\ \begin{array}{c} r \\ m-r \end{array} & \begin{pmatrix} \mathrm{id}_r & M \\ 0 & 0 \end{pmatrix} \end{array} \in F^{(r+(m-r)) \times (K_1 \uplus K_2)} \text{ where} \\ &\sharp(K_1) = r = m-1,\ M \in \mathbb{R}^{K_1 \times K_2},\ X \in \mathrm{Gl}_m(F), \\ &\Longrightarrow A|_{K_2} = A|_{K_1} M,\ A|_{K_i} = (A_{vk})_{v \in V, k \in K_i} \in \mathbb{R}^{V \times K_i},\ i = 1, 2. \end{aligned} \tag{1.112}$$

It implies that the columns $A_{-,k_1} = A(-, k_1),\ k_1 \in K_1$, are an $\mathbb{R}$-basis of the column space $A\mathbb{R}^K := \sum_{k \in K} A_{-k}\mathbb{R} \subseteq \mathbb{R}^V$ of A, and that the linear relations $A_{-k_2} = \sum_{k_1 \in K_1} A_{-k_1} M_{k_1 k_2},\ k_2 \in K_2$, hold. Notice that the Gauß algorithm and therefore the decomposition $K = K_1 \uplus K_2$ and the matrix M are not unique. This variability is essential to ensure $K_s \subseteq K_1$ or $K_s \subseteq K_2$ under

suitable conditions and to derive a suitable state space representation of $\mathcal{B}$, cf. (1.123) and Theorem 7.3.32. In the electrical engineering literature [70], [66] the branches in K_1 resp. K_2 are usually obtained as *tree* resp. *cotree (tree complement, link)* branches.
It turns out, cf. Theorem 7.3.5, that the Kirchhoff voltage resp. current law is equivalent to the equation

$$\begin{aligned} &U_{K_2} = M^\top U_{K_1} \text{ resp. } I_{K_1} = -M I_{K_2} \text{ with} \\ &U_{K_i} := (U_k)_{k\in K_i} \in \mathcal{F}^{K_i},\ I_{K_i} := (I_k)_{k\in K_i} \in \mathcal{F}^{K_i}. \end{aligned} \tag{1.113}$$

With $K_{1,s} := K_1 \cap K_s$ etc. the decompositions

$$K = K_1 \uplus K_2 = K_s \uplus K_p \text{ imply } K = K_{1,s} \uplus K_{2,s} \uplus K_{1,p} \uplus K_{2,p}. \tag{1.114}$$

Define $u := \left(\begin{smallmatrix} I_{K_{2,s}} \\ U_{K_{1,s}} \end{smallmatrix}\right) \in \mathcal{F}^{K_{2,s}\uplus K_{1,s}} = \mathcal{F}^{K_s} = \mathcal{F}^{n_s}$, and let y be the subvector of $\left(\begin{smallmatrix} U \\ I \end{smallmatrix}\right)$ that contains all components except those of u, hence $\left(\begin{smallmatrix} U \\ I \end{smallmatrix}\right) = \left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right)$ (up to the order of the components). Then the network behavior $\mathcal{B}$ can be written as

$$\begin{aligned} &\mathcal{B} := \left\{ \left(\begin{smallmatrix} U \\ I \end{smallmatrix}\right) = \left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right) \in \mathcal{F}^{K\uplus K} = \mathcal{F}^{(2n-n_s)+n_s};\ P\circ y = Q\circ u \right\} \text{ with} \\ &(P, -Q) \in \mathbb{R}[s]^{(2n-n_s)\times((2n-n_s)+n_s)},\ n := \sharp(K),\ n_s := \sharp(K_s), \end{aligned} \tag{1.115}$$

where P and Q are easily derived from M and the P_k, Q_k, cf. (7.221). Under a weak constructive condition, that is satisfied generically or almost always, $\mathcal{B}$ is an IO behavior, cf. Theorem 7.3.11, with input u and transfer matrix $H := P^{-1}Q$. Assume this. It follows the reasonable result that the source currents $I_{k_2},\ k_2 \in K_{2,s}$ and source voltages $U_{k_1},\ k_1 \in K_{1,s}$, can be chosen as input and give rise to all other branch voltages $U_k,\ k \in K\backslash K_{1,s}$, and branch currents $I_k,\ k \in K\backslash K_{2,s}$. If $\mathcal{B}$ is not an IO behavior, it has to be redesigned. The characteristic variety $\mathrm{char}(\mathcal{B}^0) = \mathrm{V}_{\mathbb{C}}(\det(P))$ of $\mathcal{B}^0 := \{y;\ P\circ y = 0\}$ can be easily determined. The steady state and superposition principle considerations concerning $\mathcal{B}$ in electrical engineering are valid if and only if the IO and asymptotic stability condition $\mathrm{V}_{\mathbb{C}}(\det(P)) \subset \mathbb{C}_-$ holds. The latter condition is often ignored, since it requires P and is hard to formulate in the usual engineering language. If it is satisfied and u has left bounded support or is periodic and y is any output to u with $P\circ y = Q\circ u$, then $y_{ss} := H\circ u$ can be identified with y (for $t\to\infty$, $y \approx y_{ss}$), and its components are all steady state branch voltages $U_k,\ k \in K\setminus K_{1,s}$, and branch currents $I_k,\ k \in K\setminus K_{2,s}$. If, additionally, H is proper and u is piecewise continuous, so is y. This is a predominant result for the analysis of electrical networks with an arbitrary number of source branches.

If $\mathcal{B}$ is an IO behavior and an n_s-port, i.e., with $2n_s$ terminals, the current which flows into the network at one terminal of a port coincides with that which flows out of it at the other terminal of the same port. This so-called *port condition* is always satisfied *for an IO n_s-port*, and need not be required as is often done in the engineering literature, cf. Corollary 7.3.12, [Wikipedia; https://en.wikipedia.org/wiki/Port (circuit theory), 24 May 2018], [66, §6.1].
Let $y_s := \left(\begin{smallmatrix} I_{K_{1,s}} \\ U_{K_{2,s}} \end{smallmatrix}\right) \in \mathcal{F}^{K_{1,s}\uplus K_{2,s}} = \mathcal{F}^{K_s} = \mathcal{F}^{n_s}$ denote the vector of complementary source currents and voltages to those of u, and define the real projection matrix C_s such that $C_s y = y_s$. Then Equation (1.32) and Theorem 7.3.18

constructively furnish a new IO behavior

$$\begin{aligned}
&\mathcal{B}_s := \begin{pmatrix} C_s & 0 \\ 0 & \mathrm{id}_{K_s} \end{pmatrix} \mathcal{B} = \{ \left(\begin{smallmatrix} C_s y \\ u \end{smallmatrix} \right); \left(\begin{smallmatrix} y \\ u \end{smallmatrix} \right) \in \mathcal{B} \} = \left\{ \left(\begin{smallmatrix} y_s \\ u \end{smallmatrix} \right) \in \mathcal{F}^{K_s \uplus K_s};\ P_s \circ y_s = Q_s \circ u \right\} \\
&\text{with } (P_s, -Q_s) \in \mathbb{R}[s]^{K_s \times (K_s \uplus K_s)} = \mathbb{R}[s]^{n_s \times (n_s + n_s)}, \\
&\qquad \mathrm{rank}(P_s) = \mathrm{rank}(P_s, -Q_s) = n_s = \sharp(K_s),\ H_s := P_s^{-1} Q_s.
\end{aligned} \tag{1.116}$$

Obviously $\mathcal{B}_s$ implies the differential system $P_s \circ y_s = Q_s \circ u$ for the source voltages and currents, and eliminates the one-port voltages and currents of the interior of the network. It is sometimes called a *black box* with the terminals as connection to the outside. Again (1.43) and (1.59) are applicable to inputs u and steady state outputs $H_s \circ u$ under the specified conditions. Define

$$\begin{aligned}
&U_s := (U_k)_{k \in K_s} \in \mathcal{F}^{K_s},\ I_s := (I_k)_{k \in K_s} \in \mathcal{F}^{K_s},\ R_s := (P_s, -Q_s) \\
&\Longrightarrow w_s := \left(\begin{smallmatrix} y_s \\ u \end{smallmatrix} \right) \left(= \left(\begin{smallmatrix} U_s \\ I_s \end{smallmatrix} \right) \text{ up to the order of the components} \right), \\
&\qquad \mathcal{B}_s = \left\{ w_s \in \mathcal{F}^{K_s \uplus K_s} = \mathcal{F}^{2n_s};\ R_s \circ w_s = 0 \right\},\ \mathrm{rank}(R_s) = n_s = \sharp(K_s).
\end{aligned} \tag{1.117}$$

Any n_s $\mathbb{R}(s)$-linearly independent columns of R_s give rise to a new IO structure of $\mathcal{B}_s$ and a new IO behavior $\widetilde{\mathcal{B}}_s$ with input $\widetilde{u} \in \mathcal{F}^{n_s}$. After the standard column and component permutations this has the form

$$\widetilde{\mathcal{B}}_s = \left\{ \widetilde{w}_s = \left(\begin{smallmatrix} \widetilde{y}_s \\ \widetilde{u} \end{smallmatrix} \right) \in \mathcal{F}^{n_s + n_s};\ \widetilde{P}_s \circ \widetilde{y}_s = \widetilde{Q}_s \circ \widetilde{u} \right\},\ \mathrm{rank}(\widetilde{P}_s) = n_s,\ \widetilde{H}_s = \widetilde{P}_s^{-1} \widetilde{Q}_s. \tag{1.118}$$

Notice that $(\widetilde{P}_s, -\widetilde{Q}_s)$ resp. $\widetilde{w}_s = \left(\begin{smallmatrix} \widetilde{y}_s \\ \widetilde{u} \end{smallmatrix} \right)$ coincide with $R_s = (P_s, -Q_s)$ resp. $w_s := \left(\begin{smallmatrix} y_s \\ u \end{smallmatrix} \right)$ up to the order of the columns resp. components, and can thus be trivially computed. Also $\mathrm{rank}(\widetilde{P}_s) = n_s$ can be easily tested. Assume this in the sequel. Then $\widetilde{\mathcal{B}}_s$ and also the original network behavior $\mathcal{B}$ are IO behaviors with input $\widetilde{u}$. Thus $\mathcal{B}$ too can be written as, cf. Theorem 7.3.21,

$$\begin{aligned}
&\widetilde{\mathcal{B}} = \left\{ \widetilde{w} = \left(\begin{smallmatrix} \widetilde{y} \\ \widetilde{u} \end{smallmatrix} \right) \in \mathcal{F}^{(2n - n_s) + n_s};\ \widetilde{P} \circ \widetilde{y} = \widetilde{Q} \circ \widetilde{u} \right\} \text{ with} \\
&(\widetilde{P}, -\widetilde{Q}) \in \mathbb{R}[s]^{(2n - n_s) \times ((2n - n_s) + n_s)},\ rank(\widetilde{P}) = 2n - n_s = n + n_p,\ \widetilde{H} = \widetilde{P}^{-1} \widetilde{Q}.
\end{aligned} \tag{1.119}$$

Again $(\widetilde{P}, -\widetilde{Q})$ resp. $\widetilde{w} = \left(\begin{smallmatrix} \widetilde{y} \\ \widetilde{u} \end{smallmatrix} \right)$ coincide with $(P, -Q)$ resp. $w = \left(\begin{smallmatrix} U \\ I \end{smallmatrix} \right) = \left(\begin{smallmatrix} y \\ u \end{smallmatrix} \right)$ up to the order of the columns resp. components and can be trivially computed, and so can be $\mathrm{char}\left(\widetilde{\mathcal{B}}^0\right) = \mathrm{V}_{\mathbb{C}}(\det(\widetilde{P}))$. Assume asymptotic stability, i.e., $\mathrm{char}\left(\widetilde{\mathcal{B}}^0\right) \subset \mathbb{C}_-$, so that steady state considerations are valid.

Simple applications of $\widetilde{H}$ furnish various forms of the *Helmholtz/Thévenin and the Mayer/Norton equivalents (theorems)*, cf. Example 7.3.16, [70, §4.2.1, §4.2.2], [66, §1.4].

Choose a period $T := 2\pi\omega^{-1} > 0$ and assume that $\widetilde{H}$ and thus $\widetilde{H}_s$ are proper. Also choose a piecewise continuous T-periodic input

$$\begin{aligned}
&\widetilde{u} = \sum_{\mu \in \mathbb{Z}} \mathbb{F}(\widetilde{u})(\mu) e^{j\mu\omega t} \text{ and define } \widetilde{y} := \widetilde{H} \circ \widetilde{u} = \sum_{\mu \in \mathbb{Z}} \widetilde{H}(j\mu\omega) \mathbb{F}(\widetilde{u})(\mu) e^{j\mu\omega t}, \\
&\widetilde{y}_s := \widetilde{H}_s \circ \widetilde{u} = \sum_{\mu \in \mathbb{Z}} \widetilde{H}_s(j\mu\omega) \mathbb{F}(\widetilde{u})(\mu) e^{j\mu\omega t},
\end{aligned} \tag{1.120}$$

where $\widetilde{y}$ resp. $\widetilde{y}_s$ are the steady state outputs of $\widetilde{\mathcal{B}}$ resp. $\widetilde{\mathcal{B}}_s$ to the input $\widetilde{u}$. According to (1.59) $\widetilde{y}$ resp. $\widetilde{y}_s$ are piecewise continuous and even continuous with uniform convergence of the Fourier series if $\widetilde{H}$ is strictly proper. Notice again that (1.120) gives an explicit Fourier series for *all* steady state voltages and currents U_k and I_k, $k \in K$, for the periodic input $\widetilde{u}$ under the condition that

$$\operatorname{rank}(P) = 2n - n_s,\ \operatorname{rank}(\widetilde{P}_s) = n_s,\ \operatorname{char}\left(\widetilde{\mathcal{B}}^0\right) \subset \mathbb{C}_-,\ \widetilde{H} \text{ proper}, \tag{1.121}$$

cf. [70, §5.5], [2, §3.3].
Assume, in particular, that $\widetilde{u} = I_s = (I_k)_{k\in K_s}$ and hence $\widetilde{y}_s = U_s = (U_k)_{k\in K_s}$. Then the matrices $\widetilde{H}_s$ resp. $\widetilde{H}_s(j\mu\omega)$ are called the *impedance transfer matrix* resp. *impedance matrix* at the frequency $\mu\omega$ of the network. If, in contrast, $\widetilde{u} = U_s$ and thus $\widetilde{y}_s = I_s$, then $\widetilde{H}_s$ resp. $\widetilde{H}_s(j\mu\omega)$, of course with a different $\widetilde{H}_s$, are the *admittance transfer matrix* resp. *admittance matrix*. For every choice of $\widetilde{u}$ *with* $\operatorname{rank}(\widetilde{P}_s) = n_s$ the corresponding matrices $\widetilde{H}_s$, $\widetilde{H}_s(j\mu\omega)$ get special names. For 2-ports with $\left(\begin{smallmatrix} U_s \\ I_s \end{smallmatrix}\right) \in \mathcal{F}^4$ there are obviously $\left(\begin{smallmatrix} 4 \\ 2 \end{smallmatrix}\right) = 6$ essentially different choices of $\widetilde{u} \in \mathcal{F}^2$. If $\operatorname{rank}(\widetilde{P}_s) = 2$, such a choice gives rise to a 2-port $\widetilde{\mathcal{B}}_s$. These various 2-ports are intensively studied in electrical engineering, cf. [66, Ch. 6].
We next explain a special state space representation of a pure RCL-network behavior $\mathcal{B}$, cf. [51], [70, §3.4], Theorem 7.3.32 and Example 7.3.33. We use a suitable Gauß algorithm to obtain decompositions $K = K_1 \uplus K_2 = K_s \uplus K_p$ with special properties. Let K_C resp. K_L denote the set of capacitance resp. inductance branches, and define

$$\begin{gathered} K_{1,C} := K_1 \cap K_C,\ K_{2,L} := K_2 \cap K_L,\ U_{1,C} := (U_k)_{k\in K_{1,C}},\ I_{2,L} := (I_k)_{k\in K_{2,L}}, \\ \widetilde{x} := \left(\begin{smallmatrix} U_{1,C} \\ I_{2,L} \end{smallmatrix}\right),\ \widetilde{u} := \left(\begin{smallmatrix} U_{1,s} \\ I_{2,s} \end{smallmatrix}\right),\ \left(\begin{smallmatrix} U \\ I \end{smallmatrix}\right) = \left(\begin{smallmatrix} \widetilde{x} \\ \widetilde{y} \\ \widetilde{u} \end{smallmatrix}\right) \text{ (up to the order of the components)} \end{gathered} \tag{1.122}$$

where, by definition, $\widetilde{y}$ contains all components of $\left(\begin{smallmatrix} U \\ I \end{smallmatrix}\right)$ that are not contained in $\widetilde{x}$ and $\widetilde{u}$. Note that the tildes have a different meaning here than in the preceding considerations. Then one can compute real matrices of suitable sizes $\widetilde{A}, \widetilde{C}, \widetilde{B}_0, \widetilde{B}_1, \widetilde{D}_0, \widetilde{D}_1$ and then $\widetilde{B} := \widetilde{B}_1 s + \widetilde{B}_0$, $\widetilde{D} := \widetilde{D}_1 s + \widetilde{D}_0$ such that the following map is a behavior isomorphism:

$$\begin{gathered} \widetilde{\mathcal{B}} := \left\{ \left(\begin{smallmatrix} \widetilde{x} \\ \widetilde{u} \end{smallmatrix}\right) \in \mathcal{F}^{\bullet+n_s};\ s \circ \widetilde{x} = \widetilde{A}\widetilde{x} + \widetilde{B} \circ \widetilde{u} \right\} \cong \mathcal{B},\ \left(\begin{smallmatrix} \widetilde{x} \\ \widetilde{u} \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} U \\ I \end{smallmatrix}\right) := \left(\begin{smallmatrix} \widetilde{x} \\ \widetilde{C}\widetilde{x} + \widetilde{D} \circ \widetilde{u} \\ \widetilde{u} \end{smallmatrix}\right), \\ \Longrightarrow \widetilde{\mathcal{B}}^0 = \left\{ \widetilde{x};\ s \circ \widetilde{x} = \widetilde{A}\widetilde{x} \right\} \cong \mathcal{B}^0,\ \widetilde{x} \mapsto \left(\begin{smallmatrix} \widetilde{x} \\ \widetilde{C}\widetilde{x} \end{smallmatrix}\right),\ \operatorname{spec}(\widetilde{A}) = \operatorname{char}(\mathcal{B}^0), \\ \widetilde{H} = (s\,\mathrm{id} - \widetilde{A})^{-1}(\widetilde{B}_1 s + \widetilde{B}_0),\ H = \left(\begin{smallmatrix} \widetilde{H} \\ \widetilde{D} + \widetilde{C}\widetilde{H} \end{smallmatrix}\right). \end{gathered} \tag{1.123}$$

Since $\widetilde{B}, \widetilde{D}$ are not constant, but $\deg_s(\widetilde{B}) \le 1$, $\deg_s(\widetilde{D}) \le 1$, this is generally not a state space representation according to Kalman, but very similar conclusions can be drawn, for instance on $\operatorname{char}(\mathcal{B}^0)$, cf. Theorem 7.3.32. The transfer matrix H is proper if and only if $\widetilde{D}_1 = 0$. Notice that the state $\widetilde{x}$ is a subvector of the trajectory $\left(\begin{smallmatrix} U \\ I \end{smallmatrix}\right)$, a rare occurrence in state space equations. If $\widetilde{B}_1 = 0, \widetilde{D}_1 = 0$, (1.123) is a state space representation according to Kalman, and especially well suited to simulate the trajectories of $\mathcal{B}$ by those of $\widetilde{\mathcal{B}}$, including initial conditions on $\widetilde{x}$ resp. $\left(\begin{smallmatrix} U \\ I \end{smallmatrix}\right)$.

We finally discuss several results on electrical power. The instantaneous power along k is $U_k(t)I_k(t)$ where the piecewise continuity of U_k and I_k is assumed. For distributions this product makes no sense. A famous theorem with a very simple proof is *Tellegen's*, cf. Theorem 7.3.36, that says that

$$\sum_{k\in K} U_k(t)I_k(t) = 0. \tag{1.124}$$

It is an energy preservation result for the total behavior with its energy sources from outside.
Assume that the network behavior $\mathcal{B}$ is an asymptotically stable IO behavior with input $\widetilde{u} = I_s := (I_k)_{k\in K_s}$, associated IO behavior $\widetilde{\mathcal{B}}_s$, output $U_s := (U_k)_{k\in K_s}$ and proper impedance transfer matrix $\widetilde{H}_s$. For a period $T = 2\pi\omega^{-1} > 0$ this implies the impedance matrix $Z := \widetilde{H}_s(j\mu\omega)$ at the frequency $\mu\omega$. Then

$$\widetilde{H}_s^\top = \widetilde{H}_s,\ Z^\top = Z, \tag{1.125}$$

i.e., these matrices are symmetric. This holds if the network consists of source and one-port branches only. These networks are called *reciprocal*, cf. [66, §6.3.1] for 2-ports.
We finally derive the average power that is supplied to the network that may be more complicated without symmetric $\widetilde{H}_s$, but is assumed asymptotically stable with proper transfer matrix. We assume a (real) piecewise continuous, periodic current vector $I_s = (I_k)_{k\in K_s}$ and the steady state voltage output $U_s := \widetilde{H}_s \circ I_s$ with the Fourier series

$$I_s = \sum_{\mu\in\mathbb{Z}} \mathbb{F}(I_s)(\mu)e^{j\mu\omega t},\ U_s = \sum_{\mu\in\mathbb{Z}} \mathbb{F}(U_s)(\mu)e^{j\mu\omega t},\ \mathbb{F}(U_s)(\mu) = \widetilde{H}_s(j\mu\omega)\mathbb{F}(I_s)(\mu). \tag{1.126}$$

The condition, that I_s is real, is equivalent to $\mathbb{F}(I_s)(-\mu) = \overline{\mathbb{F}(I_s)(\mu)}$, and likewise for $\mathbb{F}(U_s)$. The average of a piecewise continuous, T-periodic function f is defined as $T^{-1}\int_0^T f(t)dt$. Then the (average) *real power* of the sources of the network is, cf. [70, §5.5.4], [2, §3.3.4], Theorem 7.3.43,

$$\begin{aligned}\mathcal{P}_r := \sum_{k\in K_s} T^{-1}\int_0^T U_k(t)I_k(t)dt &= \mathbb{F}(I_s)(0)^\top \widetilde{H}_s(0)\mathbb{F}(I_s)(0)\\ &+ \sum_{\mu=1}^{\infty} \mathbb{F}(I_s)(\mu)^* \left(\widetilde{H}_s(j\mu\omega)^* + \widetilde{H}_s(j\mu\omega)\right)\mathbb{F}(I_s)(\mu)\end{aligned} \tag{1.127}$$

where $M^* := \overline{M^\top}$ denotes the Hermitean adjoint of a complex matrix. Notice that $\mathbb{F}(I_s)(0)$ and $\widetilde{H}_s(0)$ are real and $\widetilde{H}_s(j\mu\omega)^* + \widetilde{H}_s(j\mu\omega)$ is Hermitean, so the expression on the right is indeed real as it should be. In our approach the voltage U_k and the current I_k have the same direction for *all* $k \in K$, also for $k \in K_s$, whereas in the engineering literature U_k and I_k, $k \in K_s$, have the opposite direction. The consequence is that in our approach a negative instantaneous power $U_k(t)I_k(t)$, $k \in K_s$, means that energy flows from the source k to the interior of the network at time t, whereas a positive power means a flow towards the source. If $\mathcal{B}$ is a one-port, i.e., $\sharp(K_s) = 1$, an *apparent resp. reactive* power $\mathcal{P}_{app} \geq 0$ resp. $\mathcal{P}_{react}$ with $\mathcal{P}_{app}^2 = \mathcal{P}_r^2 + \mathcal{P}_{react}^2$ are defined and discussed, cf. [2, 3.3.4] and Corollary 7.3.45.

1.8 Stabilizing compensators

The notations and assumptions of the preceding sections remain in force, we denote $\mathcal{D} := \mathbb{C}[s]$. Chapter 9 is a variant of essential parts of Vidyasagar's book [73] and also of [19, Chs. 6,7], however with modified mathematics. We refer to [73] for the history of this approach. It deals with the synthesis of suitable behaviors, cf. the title of [73]. The chapter also owes much to Bourlès' RST-controllers [13, Ch. 6] and his suggestions for the paper [17]. Its mathematical details come from the papers [9] and [11]. The use of modules in this context is due to Quadrat [62].

The set T of asymptotically stable polynomials $t \in \mathcal{D}$, i.e., with $\mathrm{V}_{\mathbb{C}}(t) \subset \mathbb{C}_-$, is a saturated, symmetric submonoid of $\mathbb{C}[s]$, i.e., satisfies

$$1 \in T,\ 0 \notin T,\ t_1, t_2 \in T \Longleftrightarrow t_1 t_2 \in T,\ t \in T \Longleftrightarrow \bar{t} \in T. \tag{1.128}$$

The following considerations hold more generally for nonempty subsets $\Lambda_1 = \overline{\Lambda_1} \subseteq \mathbb{C}_-$ and $T = \{t \in \mathcal{D};\ \mathrm{V}_{\mathbb{C}}(t) \subseteq \Lambda_1\}$. Assume this in the sequel. For *pole placement* the set Λ_1 may be chosen finite. With today's computer algebra systems the variety $\mathrm{V}_{\mathbb{C}}(t)$ and the inclusion $\mathrm{V}_{\mathbb{C}}(t) \subseteq \Lambda_1$ can be easily determined. There are other methods (*Routh-Hurwitz criterion*) to decide $t \in T$ without computing $\mathrm{V}_{\mathbb{C}}(t)$, but these are not discussed in this book.

The monoid T gives rise to the quotient ring $\mathcal{D}_T := \left\{ft^{-1} = \frac{f}{t};\ f \in \mathcal{D}, t \in T\right\} \subseteq \mathbb{C}(s)$, that is also a principal ideal domain. Likewise, every $\mathcal{D}$-module M gives rise to the *$\mathcal{D}_T$-quotient module*

$$M_T := \left\{xt^{-1} = \frac{x}{t};\ x \in M, t \in T\right\} \text{ with } \frac{f}{t_1}\frac{x}{t_2} := \frac{fx}{t_1 t_2} \tag{1.129}$$

as scalar multiplication, and the *T-torsion submodule*

$$\begin{aligned} &\mathrm{t}_T(M) := \{x \in M;\ \exists t \in T:\ tx = 0\} = \ker\left(M \to M_T,\ x \mapsto \tfrac{x}{1}\right) \subseteq M \\ &\Longrightarrow (\mathrm{t}_T(M) = M \Longleftrightarrow M_T = 0)\,. \end{aligned} \tag{1.130}$$

Note that $\mathcal{D}$ has no zero-divisors and thus $\mathrm{t}_T(\mathcal{D}) = 0$ whereas $\mathrm{t}_T(M)$ may be nonzero. Hence the construction of M_T and the study of its properties in Section 8.1 are more difficult than those of $\mathcal{D}_T$, that are known from the construction of $\mathbb{Q}$ resp. $\mathbb{C}(s)$ from $\mathbb{Z}$ resp. $\mathcal{D} = \mathbb{C}[s]$.

Let $\mathcal{F}$ be one of the injective cogenerators $\mathrm{C}^{-\infty}$, $\mathrm{C}^{\infty}, \mathrm{t}(\mathrm{C}^{\infty}) = \oplus_{\lambda \in \mathbb{C}} \mathbb{C}[t] e^{\lambda t}$. Note that, in general in this section, t denotes a polynomial in T whereas in the formulas $e^{\lambda t}$ and $\lim_{t \to \infty}$ it denotes a time instant in $\mathbb{R}$. Then *$\mathcal{F}_T$ is an injective $\mathcal{D}_T$-cogenerator, cf. Theorem 8.3.6*, and thus gives rise to a theory of ${}_{\mathcal{D}_T}\mathcal{F}_T$-behaviors. Moreover $\mathcal{F}$ admits a $\mathcal{D}$-linear direct sum decomposition, cf. Theorem 8.3.2,

$$\begin{aligned} &\mathcal{F} := \mathrm{t}_F(\mathcal{F}) \oplus \mathcal{F}' \ni w = w_{tr} + w_{ss},\ \mathrm{t}_T(\mathcal{F}) = \oplus_{\lambda \in \Lambda_1} \mathbb{C}[t] e^{\lambda t} \subseteq \mathcal{F}_- = \oplus_{\lambda \in \mathbb{C}_-} \mathbb{C}[t] e^{\lambda t} \\ &\text{with } \mathcal{F}' \underset{\mathcal{D}}{\cong} \mathcal{F}_T,\ \widetilde{w} \mapsto \tfrac{\widetilde{w}}{1} \Longrightarrow \mathcal{F}' \underset{\text{identification}}{=} \mathcal{F}_T,\ \widetilde{w} = \tfrac{\widetilde{w}}{1}. \end{aligned} \tag{1.131}$$

The elements $t \in T$, $H \in \mathcal{D}_T$ resp. $y \in \mathrm{t}_T(\mathcal{F})$ are called T-stable polynomials, rational functions resp. signals. Due to $\Lambda_1 \subseteq \mathbb{C}_-$ T-stability implies asymptotic stability. The components w_{tr} resp. w_{ss} are again called the *transient* resp. the *steady state* of w for this decomposition. The existence of the direct summand $\mathcal{F}'$

depends on the nonconstructive *Lemma of Zorn*, and therefore neither $\mathcal{F}'$ nor, in general, the decomposition $w = w_{tr} + w_{ss}$ can be computed explicitly. However, in many situations the *unique existence* of $w = w_{tr} + w_{ss}$ is sufficient. Due to $\mathrm{t}_T(\mathcal{F}) \subseteq \mathcal{F}_-$ the limit $\lim_{t\to\infty} w_{tr}(t) = 0$ holds, i.e., for practical purposes w and w_{ss} can be identified in many situations.
If $H = \frac{f}{t_1} \in \mathcal{D}_T \subset \mathbb{C}(s)$ is a T-stable rational function and $\widetilde{u} = \frac{u}{t_2} \in \mathcal{F}' = \mathcal{F}_T$, then $H \circ \widetilde{u} = \frac{f\circ u}{t_1 t_2}$ is defined, whereas $H \circ u$ is not defined for each $u \in \mathcal{F}$, let alone for arbitrary $H \in \mathbb{C}(s)$. In [73, Chs. 3,5; (1),(2)] the meaning of Hu is not explained. Note, however, that $H \circ u$ is well-defined for $H \in \mathbb{C}(s)$ and $u \in \mathrm{C}_+^{-\infty}$.
Any behavior

$$\begin{aligned}&\mathcal{B} = \left\{w \in \mathcal{F}^l;\ R \circ w = 0\right\} \text{ with}\\ &\quad R \in \mathcal{D}^{p\times l},\ \mathrm{rank}(R) = p,\ U = \mathcal{B}^\perp = \mathcal{D}^{1\times p}R,\ M = \mathcal{D}^{1\times l}/U,\end{aligned}$$

implies

$$\begin{aligned}&\mathcal{B}_T = \left\{w \in \mathcal{F}_T^l;\ R \circ w = 0\right\} \cong \mathrm{Hom}_{\mathcal{D}_T}(M_T, \mathcal{F}_T) \text{ where}\\ &\quad U_T = \mathcal{D}_T^{1\times p}R,\ M_T = \mathcal{D}_T^{1\times l}/U_T,\\ &\mathcal{B} = \mathrm{t}_T(\mathcal{B}) \oplus \mathcal{B}_T,\ \mathrm{t}_T(\mathcal{B}) = \mathcal{B} \cap \mathrm{t}_T(\mathcal{F})^l,\ \mathcal{B}_T \underset{\mathcal{F}'=\mathcal{F}_T}{=} \mathcal{B} \cap (\mathcal{F}')^l = \mathcal{B} \cap \mathcal{F}_T^l.\end{aligned} \tag{1.132}$$

In particular, we infer the equivalence

$$\mathcal{B}_T = 0 \Longleftrightarrow M_T = 0 \Longleftrightarrow \exists t \in T \text{ with } tM = 0 \Longleftrightarrow \mathcal{B} = \mathrm{t}_T(\mathcal{B})\,(\subseteq \mathrm{t}(\mathcal{B}))\,. \tag{1.133}$$

Hence, if $\mathcal{B}_T = 0$, $\mathcal{B}$ is autonomous and called *T-autonomous*. The main application is to IO behaviors

$$\begin{aligned}&\mathcal{B} = \left\{\left(\begin{smallmatrix} y\\ u\end{smallmatrix}\right) \in \mathcal{F}^{p+m};\ P \circ y = Q \circ u\right\},\ \mathcal{B}^0 = \{y \in \mathcal{F}^p;\ P \circ y = 0\},\\ &\mathcal{B}_T = \{\left(\begin{smallmatrix} y\\ u\end{smallmatrix}\right) \in \mathcal{F}_T^p;\ P \circ y = Q \circ u\},\ \mathcal{B}_T^0 = \{y \in \mathcal{F}_T^p;\ P \circ y = 0\},\end{aligned} \tag{1.134}$$

where $(P, -Q) \in \mathcal{D}^{p\times(p+m)}$, $\mathrm{rank}(P) = p$. The *IO behavior $\mathcal{B}$ is called T-stable* if it satisfies the following equivalent conditions, cf. Theorem 8.4.2:

1. $\mathcal{B}^0$ is T-autonomous or, equivalently, $\mathcal{B}_T^0 = 0$ or $\det(P) \in T$ or $P \in \mathrm{Gl}_p(\mathcal{D}_T)$.
2. (i) $\mathcal{B}_T$ is controllable or M_T is $\mathcal{D}_T$-free or $\mathrm{char}(\mathcal{B}) \subseteq \Lambda_1$.
   (ii) H is T-stable, i.e., $H \in \mathcal{D}_T^{p\times m}$.

Assume that $\mathcal{B}$ is T-stable. This, ${}_{\mathcal{D}_T}\mathcal{F}_T$ and $H = P^{-1}Q$ imply

$$\begin{aligned}\mathcal{B}_T &= \left\{\left(\begin{smallmatrix} y\\ u\end{smallmatrix}\right) \in \mathcal{F}_T^{p+m} = (\mathcal{F}')^{p+m};\ P \circ y = Q \circ u\right\}\\ &= \left\{\left(\begin{smallmatrix} y\\ u\end{smallmatrix}\right) \in \mathcal{F}_T^{p+m} = (\mathcal{F}')^{p+m};\ y = H \circ u\right\}\\ &\Longrightarrow \forall \left(\begin{smallmatrix} y\\ u\end{smallmatrix}\right) \in \mathcal{B} \text{ with } u = u_{tr} + u_{ss}:\ y = y_{tr} + y_{ss},\ y_{ss} = H \circ u_{ss}.\end{aligned} \tag{1.135}$$

Hence $H \circ u_{ss}$ is the steady state of y, cf. (1.131) and (1.135). The latter equation is the main tool to simplify equations for T-stable IO behaviors.
We have seen that properness of a transfer matrix is an important property. For T-stable IO behaviors this means

$$H \in \mathbb{C}[s]_T^{p\times m} \cap \mathbb{C}(s)_{\mathrm{pr}}^{p\times m} = \mathcal{S}^{p\times m},\ \mathcal{S} := \mathbb{C}[s]_T \cap \mathbb{C}(s)_{\mathrm{pr}} \subset \mathbb{C}(s), \tag{1.136}$$

where $\mathbb{C}(s)_{\mathrm{pr}}$ resp. $\mathcal{S}$ is the ring of proper resp. of proper and T-stable rational functions. The computations in [73] essentially use that $\mathcal{S}$ is euclidean. Instead, we choose $\alpha \in \Lambda_1$, define the variable $\widehat{s} := (s-\alpha)^{-1} \in \mathbb{C}(s)$ and show in Theorem 8.5.4 that

$$\mathcal{S} = \mathbb{C}[\widehat{s}]_{\widehat{T}} \subset \mathbb{C}(s),\ \widehat{T} := \left\{ \widehat{t} := t\widehat{s}^{\deg_s(t)} = \tfrac{t}{(s-\alpha)^{\deg_s(t)}};\ t \in T \right\} \tag{1.137}$$

where $\mathbb{C}[\widehat{s}]$ is the polynomial algebra in the variable $\widehat{s}$, cf. [59]. Computations with this quotient ring $\mathcal{S} = \mathbb{C}[\widehat{s}]_{\widehat{T}}$ of a polynomial algebra are as simple as with the polynomial algebra itself, are implemented in every *Computer Algebra* system, and simpler than those in a general euclidean ring, e.g., in $\mathcal{S}$. In Chapter 10 the ring $\mathcal{S}$ is used for the construction of *functional observers*.
A feedback compensator or controller $\mathcal{B}_2$ of a *plant* $\mathcal{B}_1$ is used to stabilize the plant and to steer its output into a desired direction. The mathematical model is given by two IO behaviors

$$\begin{aligned} \mathcal{B}_1 &:= \left\{ \left(\begin{smallmatrix} y_1 \\ u_1 \end{smallmatrix}\right) \in \mathcal{F}^{p+m};\ P_1 \circ y_1 = Q_1 \circ u_1 \right\},\ (P_1, -Q_1) \in \mathcal{D}^{p\times(p+m)},\ \operatorname{rank}(P_1) = p, \\ \mathcal{B}_2 &:= \left\{ \left(\begin{smallmatrix} u_2 \\ y_2 \end{smallmatrix}\right) \in \mathcal{F}^{p+m};\ P_2 \circ y_2 = Q_2 \circ u_2 \right\},\ (-Q_2, P_2) \in \mathcal{D}^{m\times(p+m)},\ \operatorname{rank}(P_2) = m. \end{aligned} \tag{1.138}$$

We use $\left(\begin{smallmatrix} u_2 \\ y_2 \end{smallmatrix}\right)$ instead of $\left(\begin{smallmatrix} y_2 \\ u_2 \end{smallmatrix}\right)$ for dimension reasons since the output (input) y_1 (u_1) of the *plant* $\mathcal{B}_1$ is assumed to have the same dimension p (m) as the input (output) u_2 (y_2) of the *compensator* $\mathcal{B}_2$. *Feedback* means to add (feed back) the output of $\mathcal{B}_1$ ($\mathcal{B}_2$) to the input of $\mathcal{B}_2$ ($\mathcal{B}_1$). Define $y := \left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right)$, $u := \left(\begin{smallmatrix} u_2 \\ u_1 \end{smallmatrix}\right) \in \mathcal{F}^{p+m}$. Then the feedback interconnection of the two behaviors is the behavior

$$\begin{aligned} \mathcal{B} := \operatorname{fb}(\mathcal{B}_1, \mathcal{B}_2) &:= \left\{ \left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right) \in \mathcal{F}^{2(p+m)};\ \left(\begin{smallmatrix} y_1 \\ u_1+y_2 \end{smallmatrix}\right) \in \mathcal{B}_1,\ \left(\begin{smallmatrix} y_2 \\ u_2+y_1 \end{smallmatrix}\right) \in \mathcal{B}_2 \right\} \\ &= \left\{ \left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right) \in \mathcal{F}^{2(p+m)};\ P \circ y = Q \circ u \right\} \text{ where} \\ P := \left(\begin{smallmatrix} P_1 & -Q_1 \\ -Q_2 & P_2 \end{smallmatrix}\right),\ Q &:= \left(\begin{smallmatrix} 0 & Q_1 \\ Q_2 & 0 \end{smallmatrix}\right) \in \mathcal{D}^{(p+m)\times(p+m)}. \end{aligned} \tag{1.139}$$

If $\operatorname{rank}(P) = p+m$ or $\det(P) \neq 0$, this is an IO behavior with transfer matrix $H = P^{-1}Q$. One then says that *the feedback behavior is well-posed*, and calls $H = P^{-1}Q$ the *closed loop transfer matrix*. Assume this in the sequel.
The first goal is the *T-stabilization* of $\mathcal{B}_1$ by $\mathcal{B}_2$, i.e., the T-stability, especially asymptotic stability, of $\mathcal{B} = \operatorname{fb}(\mathcal{B}_1, \mathcal{B}_2)$. This means $\det(P) \in T$ or that $\mathcal{B}_T$ is controllable and $H = P^{-1}Q \in \mathcal{D}_T^{(p+m)\times(p+m)}$. Then (1.135) is applicable to $\operatorname{fb}(\mathcal{B}_1, \mathcal{B}_2)$.
For a given plant $\mathcal{B}_1$ a compensator $\mathcal{B}_2$ with well-posed and T-stable $\mathcal{B} = \operatorname{fb}(\mathcal{B}_1, \mathcal{B}_2)$ exists if and only if $\mathcal{B}_{1,T}$ is controllable or $\operatorname{char}(\mathcal{B}_1) \subseteq \Lambda_1$, and one then says that $\mathcal{B}_1$ is *T-stabilizable* and $\mathcal{B}_2$ is a *T-stabilizing compensator*, cf. Corollary 9.1.11. This is the main use of controllability in this book. Assume the T-stabilizability of $\mathcal{B}_1$ in the sequel.
In Theorem 9.1.10 we construct, for T-stabilizable $\mathcal{B}_1$, *all* controllable compensators $\mathcal{B}_2$ such that $\mathcal{B} = \operatorname{fb}(\mathcal{B}_1, \mathcal{B}_2)$ is well-posed and T-stable. These $\mathcal{B}_2$ depend on T-stable rational $m \times p$-matrices X as *parameter*, and therefore one talks about the *parametrization* of these compensators. In Theorem 9.1.20 we construct (parametrize) all compensators for which, in addition, the closed loop transfer matrix H is proper, i.e., $H \in \mathcal{S}^{(p+m)\times(p+m)}$, cf. (1.137). Finally, in Theorem 9.1.34, all $\mathcal{B}_2$ with, in addition, proper transfer matrix $H_2 = P_2^{-1}Q_2$,

are parametrized. We do not assume that the transfer matrix $H_1 = P_1^{-1}Q_1$ of $\mathcal{B}_1$ is proper, but most real plants have this property. The properness of H_2 is important since it enables a state space representation of $\mathcal{B}_2$ according to Kalman and its construction with elementary building blocks. If also $\mathcal{B}_1$ is a state space behavior, then the preceding considerations yield many more stabilizing compensators than those, that are usually constructed by means of *Luenberger state observers* connected with *state feedback*, cf. [39, p. 523], [60, §10.5-6], Sections 9.1.5 and 11.5.

The preceding theory is applicable to finite $\Lambda_1 \subset \mathbb{C}_-$. For the T-stability condition $\mathrm{char}(\mathcal{B}^0) \subseteq \Lambda_1$ one says that the *poles of* $\mathcal{B}^0$ *have been placed into or assigned to* Λ_1 , cf. Theorem 9.1.26.

As explained in the preceding sections stability is a necessary condition for almost any IO behavior. But stabilization is not a goal for itself except in special cases, for instance to stabilize a building after an earthquake. Switching off an asymptotically stable electrical network stabilizes it, but the network cannot serve any further purpose. *Control design* in Section 9.2 is devoted to the choice of compensators, among those just described, that serve a useful purpose. We always assume that the feedback behavior and the T-stabilizing compensator have proper transfer matrices. The main treated design problem is *tracking and disturbance rejection*. In this case a nonzero polynomial ϕ and three signals $r, u_2 \in \mathcal{F}^p,\ u_1 \in \mathcal{F}^m$, with $\phi \circ (u_2, u_1, r) = (0,0,0)$ are considered where r is a given *reference signal* and u_1 resp. u_2 are unknown *disturbance signals* of the input resp. output of $\mathcal{B}_1$. These signals are, of course, not assumed T-stable, i.e., $\phi \notin T$ in general. The input of the compensator is y_1+u_2-r where y_1+u_2 is the actual disturbed output of the plant in the feedback behavior. For instance, $\phi = s\ (s^2, s^3)$ means that the signals are constant (linear, quadratic) functions of t. The design goal is to construct $\mathcal{B}_2$ such that $y_1 + u_2 - r$ is T-stable, in particular, $\lim_{t\to\infty}((y_1+u_2)-r)(t) = 0$, i.e., the actual disturbed output signal of the plant in the feedback behavior $\mathcal{B}$ coincides, for practical purposes, with the desired reference signal. One says that in $\mathcal{B}$ *the output of the plant* tracks *the reference signal and rejects the disturbance signals.*

The polynomial ϕ is fixed and thus restricts the admissible unknown disturbance signals, but generically (almost) all ϕ can be chosen for a given plant. We derive a necessary and sufficient condition for the existence of such compensators $\mathcal{B}_2$ for given plant $\mathcal{B}_1$ and polynomial ϕ, and parametrize all these. The most important constructive results are Theorems 9.2.8, 9.2.11 and 9.2.17 and the algorithm in Corollary 9.2.10. In Section 9.2.2 we also discuss the significance of the *(transmission) zeros* of the plant's transfer matrix in this context. All matrix computations require the Smith form of polynomial matrices only.

In the literature the reference signal r and likewise the disturbance signals are often assumed, cf. [19, (17), p. 198], [21, (9-100), (9-101), p. 495], [73, §7.5], as

$$\begin{aligned} &\widetilde{r} = H_r \circ \delta = \mathcal{L}^{-1}(H_r) = \alpha_r Y,\ \alpha_r \in \mathrm{t}(\mathcal{F})^p \text{ where} \\ &H_r = \mathcal{L}(\widetilde{r}) = \phi^{-1}Q_r,\ Q_r \in \mathbb{C}[s]^p,\ \deg_s(\phi) > \deg_s(Q_r) \Longrightarrow \phi \circ \alpha_r = 0. \end{aligned} \tag{1.140}$$

Conversely, if

$$\begin{aligned} &\phi \circ r = 0,\ \text{then } \phi \circ (rY) = (\phi \circ r)Y + Q_r \circ \delta = Q_r \circ \delta,\ Q_r \in \mathbb{C}[s]^p, \\ &\Longrightarrow \widetilde{r} := rY = H_r \circ \delta,\ H_r := \phi^{-1}Q_r,\ \deg_s(\phi) > \deg_s(Q_r),\ \widetilde{r}|_{[0,\infty)} = r|_{[0,\infty)}. \end{aligned} \tag{1.141}$$

For $(\widetilde{u}_2, \widetilde{u}_1, \widetilde{r})$ instead of (u_2, u_1, r) the design goal is $e = y_1 + \widetilde{u}_2 - \widetilde{r} = H_e \circ \delta$ with strictly proper and T-stable $H_e \in \mathbb{C}(s)^p$, cf. [19, p. 206, (70)], [73, p. 296, (R2)]. In Remark 9.2.6 we show that our theory furnishes such an H_e. Laplace transform techniques in the quoted books require these different reference and disturbance signals.
For the preceding data assume that $\mathcal{B}_2$ is a compensator for $\mathcal{B}_1$ and that $\widetilde{\mathcal{B}}_1 \subseteq \mathcal{F}^{p+m}$ is another IO plant. One says that $\mathcal{B}_2$ is a *robust compensator for* $\mathcal{B}_1$ if it is also a compensator (with all properties from above) for all *T-stabilizable* $\widetilde{\mathcal{B}}_1$ *near* $\mathcal{B}_1$ or all controllable $\widetilde{\mathcal{B}}_{1,T}$ near $\mathcal{B}_{1,T}$. This requires a topology on the set of controllable IO behaviors $\widetilde{\mathcal{B}}_{1,T} \subseteq \mathcal{F}_T^{p+m}$, cf. Corollary and Definition 9.2.30. The main result is proven only for the case $\Lambda_1 = \mathbb{C}_-$ where T- and asymptotic stability coincide. Two such topologies are derived from a norm $\|H\|$ and a finer norm $\|H\|_1$ on the algebra $\mathcal{S}$ of T-stable rational functions, cf. Theorem 9.2.20 and Remark 9.2.49, given for $H = H_0 + H_{\text{spr}} \in \mathcal{S}$ with $H_0 \in \mathbb{C}$ and strictly proper H_{spr} by

$$\begin{aligned} \|H\| &:= \sup_{\omega \in \mathbb{R}} |H(j\omega)| = \sup_{\omega \in \mathbb{R}} |H_0 + H_{\text{spr}}(j\omega)| \\ \|H\|_1 &:= |H_0| + \|h\|_1,\ h := H_{\text{spr}} \circ \delta,\ \|h\|_1 := \int_0^\infty |h(t)| dt,\ \|H\| \le \|H\|_1. \end{aligned} \tag{1.142}$$

These norms are naturally extended to matrix norms $\|H\|$ and $\|H\|_1$. A matrix $H = H_0 + H_{\text{spr}} \in \mathcal{S}^{p+m}$ with $h := H_{\text{spr}} \circ \delta \in (\mathrm{L}_+^1)^{p+m}$ induces the operators

$$\begin{array}{lcl} H\circ : & (\mathrm{L}_+^2)^{p+m} \longrightarrow & (\mathrm{L}_+^2)^{p+m} \\ & \cap & \cap \\ H\circ : & (\mathfrak{L}^2)^{p+m} \longrightarrow & (\mathfrak{L}^2)^{p+m} \end{array} \tag{1.143}$$

and

$$\begin{array}{lcl} H\circ : & (\mathrm{L}^\infty)^{p+m} \longrightarrow & (\mathrm{L}^\infty)^{p+m} \\ & \cap & \cap \\ H\circ : & (\mathfrak{L}^\infty)^{p+m} \longrightarrow & (\mathfrak{L}^\infty)^{p+m} \end{array} \tag{1.144}$$

where $\mathfrak{L}^2$ is the Banach completion of L_+^2 and of L^2 and $\mathfrak{L}^\infty$ that of L^∞. These operators, in turn, have the finite norms, cf. Theorems 9.2.35 and 9.2.45,

$$\begin{aligned} \|H \circ\|_2 &:= \|H\circ :\ \left(\mathfrak{L}^2\right)^{p+m} \to \left(\mathfrak{L}^2\right)^{p+m}\| \\ &= \|H\| := \sup_{\omega \in \mathbb{R}} \sigma(H(j\omega)),\ j := \sqrt{-1}, \\ \|H \circ\|_\infty &:= \|H\circ :\ (\mathfrak{L}^\infty)^{p+m} \to (\mathfrak{L}^\infty)^{p+m}\| \\ &= \|H\|_1 := \max_{i=1,\cdots,p} \sum_{j=1}^m (|H_{0,ij}| + \|h_{ij}\|_1) \end{aligned} \tag{1.145}$$

where $\sigma(A) = \|A\|_2$ denotes the largest singular value of a complex matrix A, cf. [25], [77, p. 107], [13, p. 518]. In the literature robust control with the topology derived from $\|H\|$ is called *H_∞-control*, and $\|H\|$ is denoted as $\|H\|_\infty$ although this norm refers to the Hilbert space $\mathfrak{L}^2$ with its norm $\| - \|_2$. We do not know of a standard terminology for robust control derived from $\|H\|_1$.
Theorem 9.2.50 is the main robustness result with two assertions:

(i) *For* $\Lambda_1 = \mathbb{C}_-$ *the constructed compensators are robust with respect to both derived topologies.*
(ii) Let $\widetilde{\mathcal{B}}_1$ be near $\mathcal{B}_1$ in one of these topologies, and let $H \in \mathcal{S}^{(p+m)\times(p+m)}$ resp. $\widetilde{H} \in \mathcal{S}^{(p+m)\times(p+m)}$ be the asymptotically stable closed loop transfer matrices of $\mathcal{B} := \mathrm{fb}(\mathcal{B}_1, \mathcal{B}_2)$ resp. $\widetilde{\mathcal{B}} = \mathrm{fb}(\widetilde{\mathcal{B}}_1, \mathcal{B}_2)$. We show in Theorems 9.2.47 and 9.2.50 that

(a) If $\lim \widetilde{\mathcal{B}}_1 = \mathcal{B}_1$ in the $\| - \|$-topology, derived from that of $\mathcal{S}$, then

$$\lim \|\widetilde{H} - H\| = \lim \|\widetilde{H} \circ - H \circ\|_2 = 0. \tag{1.146}$$

(b) If $\lim \widetilde{\mathcal{B}}_1 = \mathcal{B}_1$ in the $\| - \|_1$-topology, derived from that of $\mathcal{S}$, then

$$\lim \|\widetilde{H} - H\|_1 = \lim \|\widetilde{H} \circ - H \circ\|_\infty = 0. \tag{1.147}$$

In words, assertion (b) ((a) analogous) reads: *If the plant* $\widetilde{\mathcal{B}}_1$ *is near* $\mathcal{B}_1$ *in the* $\| - \|_1$*-topology, then the BIBO stable transfer operator* $\widetilde{H}\circ$ *exists and is near* $H\circ$ *in the* $\| - \|_\infty$*-norm.* In other words: *The BIBO stable transfer operator* $\widetilde{H}\circ$ *depends continuously on the plant* $\widetilde{\mathcal{B}}_1$.
The robustness properties (i) and (ii) of the compensator are required due to model uncertainty*, cf. [77, Ch. 9], i.e., that for various reasons the data of the plant model deviate slightly from those of the real modeled plant.*
The details for the preceding assertions are relatively difficult, but are also completely proved. In particular, continuity properties of the *Fourier integral (transform)*, denoted by the same letter as the Fourier series,

$$\mathbb{F} : \mathrm{L}^1_t \to \mathrm{C}^0_\omega,\ u \mapsto \mathbb{F}(u),\ \mathbb{F}(u)(\omega) = \int_{-\infty}^{\infty} u(t) e^{-j\omega t} dt, \tag{1.148}$$

and its extensions have to be used. These, in turn, imply continuity properties of the *Laplace transform*

$$\begin{aligned} &\mathcal{L} : \mathrm{L}^1_{\geq 0} := \left\{u \in \mathrm{L}^1;\ \mathrm{supp}(u) \subseteq [0, \infty)\right\} \to \mathrm{C}^0,\ u \mapsto \mathcal{L}(u),\ \text{with} \\ &\mathcal{L}(u)(s) := \int_0^\infty u(t) e^{-st} dt,\ s \in \mathbb{C},\ \Re(s) \geq 0, \end{aligned} \tag{1.149}$$

that extends that from (1.40) on $Y\mathcal{F}_-$, cf. (1.101). We do not need or use the Fourier transform of general temperate distributions [37, Thm. 7.1.10].

1.9 Further systems theories in this book

The theory of this book is applicable to all situations where a principal ideal operator domain $\mathcal{D}$ and an injective cogenerator signal module ${}_{\mathcal{D}}\mathcal{F}$ are given, and where linear systems $R \circ w = x$ and behaviors $\left\{w \in \mathcal{F}^l;\ R \circ w = 0\right\}$ are of interest. In most cases $\mathcal{D}$ is a polynomial algebra $\mathcal{D} = F[s]$ over a field F, but consider ${}_{\mathcal{D}_T}\mathcal{F}_T$ from (1.131) where the operator domain is not polynomial.
The theory of the preceding sections is also valid for the real base field, the injective cogenerators being

$${}_{\mathbb{R}[s]}\mathrm{C}^\infty(\mathbb{R}, \mathbb{R}) \subset_{\mathbb{C}[s]} \mathrm{C}^\infty(\mathbb{R}, \mathbb{C}),\ {}_{\mathbb{R}[s]}\mathrm{C}^{-\infty}(\mathbb{R}, \mathbb{R}) \subset_{\mathbb{C}[s]} \mathrm{C}^{-\infty}(\mathbb{R}, \mathbb{C}). \tag{1.150}$$

Since real behaviors are the real parts of complex ones, most results can be directly transferred from the complex to the real case. Few real results require additional considerations, and these are carried out in detail.
The *LTI (linear time-invariant)-discrete-time-behaviors* in this book for an arbitrary base field F use the injective cogenerator

$$_{F[s]}F^{\mathbb{N}} \ni w = (w(0), w(1), w(2), \cdots),\ (s \circ w)(t) = w(t+1), \tag{1.151}$$

of sequences in F. For any matrix $R = (R_{\alpha\beta})_{\alpha,\beta} \in F[s]^{k\times l}$ with $R_{\alpha\beta} = \sum_{\mu\in\mathbb{N}} R_{\alpha\beta,\mu} s^{\mu}$ an inhomogeneous system has the form

$$\begin{aligned} &R \circ w = x,\ w \in \left(F^{\mathbb{N}}\right)^{l},\ x \in \left(F^{\mathbb{N}}\right)^{k},\ \text{or} \\ &\forall t \in \mathbb{N},\ \forall \alpha = 1, \cdots, k:\ \sum_{\mu\in\mathbb{N}} \sum_{\beta=1}^{l} R_{\alpha\beta,\mu} w_{\beta}(t+\mu) = x_{\alpha}(t). \end{aligned} \tag{1.152}$$

So the basic equations are linear systems of *difference equations*, a famous one being the *Fibonacci equation*

$$\begin{aligned} &(s^2 - s - 1) \circ w = 0 \Longleftrightarrow \forall t \in \mathbb{N}:\ w(t+2) = w(t+1) + w(t) \\ &\text{with } w(0) := 1,\ w(1) := 1,\ \text{hence } w = (1, 1, 2, 3, 5, 8, \cdots) \in \mathbb{R}^{\mathbb{N}}. \end{aligned} \tag{1.153}$$

A variant of this theory is furnished by the injective cogenerator

$$\begin{aligned} &_{\mathcal{D}}F^{\mathbb{Z}} \ni w = (\cdots, w(-2), w(-1), w(0), w(1), w(2), \cdots),\ (s \circ w)(t) = w(t+1), \\ &\text{where } \mathcal{D} := F[s, s^{-1}] = \oplus_{\mu\in\mathbb{Z}} F s^{\mu} = \{\text{Laurent polynomials}\}. \end{aligned} \tag{1.154}$$

The theory is almost the same as that for (1.151), but not discussed in this book.
Over the base fields $\mathbb{C}$ and $\mathbb{R}$ almost all results of the continuous-time theory have a discrete-time analogue, in particular those of Chapter 9 on stabilizing compensators, with the exception of $\lim \|\widetilde{H} \circ - H \circ\|_2 = 0$ in (1.146) and $\lim \|\widetilde{H} \circ - H \circ\|_{\infty} = 0$ in (1.147). These analogues can be derived, but we have not done this. Most proofs for the two cases, in particular those of Chapter 9, are carried out simultaneously for $_{F[s]}\mathcal{F}$-behaviors with $F = \mathbb{C}, \mathbb{R}$ and injective cogenerators $\mathcal{F} = \mathrm{C}^{-\infty}(\mathbb{R}, F)$ or $\mathcal{F} = F^{\mathbb{N}}$.
The paper [10] studies more general feedback interconnections and quotes the corresponding literature, cf. [60, §10.8.2].

1.10 Additional results

Here we mention additional results with new derivations that are, however, not further used in the book.
In Section 5.3.2 we explain the connection of the behavioral and the Rosenbrock languages with that of the *French school* of Fliess, Bourlès [13] et al..
Section 7.2.9 is devoted to a short explanation of *Mikusinski's calculus* that is used as an alternative for one-dimensional distribution theory, for instance by Fliess, but not in this book.
For nonproper $H \in \mathbb{C}(s)^{p\times m}$ there are inputs $u \in (\mathrm{C}^{\infty})^{m}\, Y$ with a jump at

$t = 0$ such that $y := H \circ u$ has *impulsive components* in $\mathbb{C}[s]^p \circ \delta$. In Section 7.2.10 we compute these components by a modification of Bourlès' method in [12], cf. also [72, §4.2] and [5].
In Section 11.5 we construct compensators for state space behaviors by means of *Luenberger state observers and state feedback*, cf. [47], [60, §10.5, §10.6]. This construction method is very special and does not furnish all possible compensators, but was historically the first.
In *model matching*, cf. Section 9.2.6, one constructs a compensator that realizes a given proper and T-stable transfer matrix H_{y_1,u_2} from u_2 to y_1 of the closed loop behavior.
In Chapter 10 we construct and parametrize so-called *functional T-observers*. These observers were studied by many colleagues, in particular intensively by Fuhrmann [31], and were applied for the construction of compensators, but in Chapter 9 they are not needed or used.

1.11 System theories not discussed in this book

For an obvious reason the following remarks are very short in those areas that we have not studied ourselves. Of course, this is no statement whatsoever on the relative importance of the areas and of the researchers' contributions.

1. *Multidimensional systems*: The multivariate polynomial algebra $\mathbb{C}[s]$ with $s := (s_1, \cdots, s_n)$, $n > 1$, acts on $u = u(x_1, \cdots, x_n) \in \mathrm{C}^\infty(\mathbb{R}^n, \mathbb{C})$ or, more generally, on $u \in \mathcal{D}'(\mathbb{R}^n, \mathbb{C})$ (Schwartz' distributions) by partial differentiation, $s_i \circ u = \partial u / \partial x_i$, and makes these signal spaces injective cogenerators. As mentioned, the injectivity was shown by very difficult work [26], [58], [49] whose usefulness for systems theory was established in [53]. There is also an analogous theory for difference equations [53]. The corresponding behaviors are called *multidimensional*, for which Theorem 2.3.18 holds. Many authors have contributed to this field in the last decades, among them Bisiacco, Bose, Fornasini, Kaczorek, Lin, Marchesini, Owens, Pommaret, Quadrat, Robertz, Rocha, Rogers, Shankar, Valcher, Willems, Wood, Zampieri, Zerz and also the authors of this book. In general, these authors have not considered systems with additional boundary conditions.

2. *Infinite-dimensional systems*: There is a different, very important and vast multidimensional theory of partial differential equations with boundary conditions, cf. [24]. An outstanding author in this area was J.-L. Lions. We have not studied this field.

3. *LTV (linear time-varying) state space systems*: Many advanced results on differential and difference systems

$$\begin{aligned} &dx/dt = A(t)x(t) + B(t)u(t),\ y(t) = C(t)x(t) + D(t)u(t) \ \text{(continuous-time)},\\ &x(t+1) = A(t)x(t) + B(t)u(t),\ y(t) = C(t)x(t) + D(t)u(t) \ \ \text{(discrete-time)}, \end{aligned} \tag{1.155}$$

and surveys of the literature are contained in the books [64], [35], [36] and, partially, in the other cited textbooks, cf. [21], [20, Ch. 2], [3, §2.6]. Recently Anderson, Berger, Hill, Ilchmann, Wirth have contributed to this area.

4. *The behavioral theory of implicit LTV differential systems (difference systems)*: This is more difficult than 3. for two reasons:
(i) The domain $\mathcal{D}$ contains the operators $f = \sum_{\mu \in \mathbb{N}} f_\mu s^\mu$ with functions (instead of constants) $f_\mu(t)$. Again f acts on u via $f \circ u = \sum_\mu f_\mu u^{(\mu)}$. The domain $\mathcal{D}$ is noncommutative since $sf_\mu = f_\mu s + f'_\mu$ (for differential equations). Its algebraic properties depend very much on the choice of the coefficient functions.
(ii) The choice of a suitable signal module and the proof of its injectivity is not obvious.
We refer to the papers [29], [14], [15], [16], [18], [56], [57] for various solutions and references to the literature. Schmale, Ilchmann, Mehrmann, Rocha, Zerz have recently contributed to this field.

5. *Algebraic Analysis*: Many outstanding mathematicians have contributed to this area, i.e., the algebraic theory of *noncommutative noetherian domains of partial differential operators with variable coefficients* [8], among them Hörmander, Kashiwara, Malgrange, Pommaret, Sato. More recently, the French school, in particular Quadrat, Robertz and many other researchers have developed its computational side. Corresponding behaviors, i.e., solution spaces, have not been studied from the engineering point of view, but see [61]. The area of partial differential equations is, of course, one of the largest in mathematics.

6. *Convolution behaviors*: These use the signal module $\mathcal{E} := \mathrm{C}^\infty(\mathbb{R}, \mathbb{C})$ as in this book, but the larger commutative, but nonnoetherian operator domain $\mathcal{E}'$ of all distributions with compact support with the convolution multiplication that acts on $\mathcal{E}$ by convolution. A typical case is the *delay-differential equation*

$$((\delta' - \delta_1) * y)(t) = y'(t) - y(t-1) = u(t). \tag{1.156}$$

The functions in torsion behaviors $\mathcal{B} := \{y \in \mathcal{E};\ T * y = 0\}$, $0 \neq T \in \mathcal{E}'$, are called *mean-periodic* and were studied by many outstanding analysts. The paper [17] on convolution behaviors also discusses the relevant literature and the principal contributors, among them Schwartz, Ehrenpreis, Berenstein, Glüsing-Lürssen, Zampieri.

7. *Nonlinear systems*: These are mostly described by state space equations $x' = f(x, u)$, $y = g(x, u)$, where $x(t) \in \mathbb{R}^n$ is the state at time t and $u(t) \in \mathbb{R}^m$ the control. Most real systems are originally nonlinear. One important solution method is *linearization*, i.e., the approximation of the nonlinear system by a linear one. See [68] for a broad discussion.

8. *Optimal control*: In context with Chapter 9 this means the choice of a stabilizing compensator among all parametrized ones that, for instance, optimizes a chosen cost function. We refer to [77], [73, Ch. 6], [36, Ch. 9].

9. *Stochastic systems theory*: This is used, for instance, to replace the very restricted disturbance signals u_1, u_2 with $\phi \circ u_i = 0$ for given ϕ by wider classes of signals with special probability distributions. We refer to [13, Ch. 11] for an introduction.

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