Proseminar „Einführung in die Mathematik 1"
WS 2010/11
16. Dezember 2011

62) Eine Eigenbasis einer Matrix $A \in K^{n \times n}$ ist eine Basis von $K^{n \times 1}$, die aus Eigenvektoren von A besteht.

$A := \begin{pmatrix} -2 & -6 \\ 2 & 5 \end{pmatrix}$. Charakteristisches Polynom von A: $\det(xI_2 - A) = \det\begin{pmatrix} x+2 & 6 \\ -2 & x-5 \end{pmatrix} =$
$= (x+2)(x-5) - 6\cdot(-2) = x^2 + 2x - 5x - 10 + 12 = x^2 - 3x + 2.$

$\det(xI_2 - A) = 0 \Leftrightarrow x^2 - 3x + 2 = 0 \Leftrightarrow$
$x = \frac{3}{2} + \sqrt{\frac{9}{4} - \cancel{4}2}$ oder $x = \frac{3}{2} - \sqrt{\frac{9}{4} - \cancel{4}2} \Leftrightarrow$
$x = 2$ oder $x = 1$

<u>Eigenwert $x_1 = 2$:</u> $x_1 I_2 - A = \begin{pmatrix} 4 & \boxed{6} \\ -2 & -3 \end{pmatrix}$.

$E(A, x_1) = L(x_1 I_2 - A) = \mathbb{R}\begin{pmatrix} -3 \\ 2 \end{pmatrix} = \mathbb{R}\begin{pmatrix} \boxed{-6} \\ 4 \end{pmatrix}$

Eigenwert $\lambda_2 = 1$: $\lambda_2 I_2 - A = \begin{pmatrix} 3 & 6 \\ -2 & -4 \end{pmatrix}$

$$E(A, \lambda_2) = L(\lambda_2 I_2 - A) = \mathbb{R}\begin{pmatrix} -6 \\ 3 \end{pmatrix} = \mathbb{R}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Also gilt: $S = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}$,

$$AS = \begin{pmatrix} -2 & -6 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -6 & -2 \\ 4 & 1 \end{pmatrix},$$

$$\left(\begin{array}{cc|cc} -3 & -2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 2 & 1 & 0 & 1 \end{array}\right) \to$$

$$\to \left(\begin{array}{cc|cc} 1 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & \frac{2}{3} & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & -2 & -3 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & -3 \end{array}\right)$$

Also: $S^{-1} = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}$, $S^{-1}AS = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} -6 & -2 \\ 4 & 1 \end{pmatrix} =$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

$B := \begin{pmatrix} \frac{11}{9} & \frac{80}{9} \\ \frac{4}{9} & \frac{7}{9} \end{pmatrix} \in \mathbb{R}^{2\times 2}$. $\det(\lambda I_2 - B) = \det\begin{pmatrix} \lambda - \frac{11}{9} & -\frac{80}{9} \\ -\frac{4}{9} & \lambda - \frac{7}{9} \end{pmatrix}$

$$= \left(\lambda - \frac{11}{9}\right)\left(\lambda - \frac{7}{9}\right) - \left(-\frac{80}{9}\right)\left(-\frac{4}{9}\right) = \lambda^2 - 2\lambda - \frac{243}{81} =$$

$$= c^2 - 2c - 3.$$

$$\det(c I_2 - B) = 0 \iff c^2 - 2c - 3 = 0 \iff$$
$$c_{1,2} = 1 \pm \sqrt{1^2+3} = \begin{cases} 3 \\ -1 \end{cases}$$

<u>Eigenwert $c_1 = 3$:</u> $c_1 I_2 - B = \begin{pmatrix} \frac{16}{9} & -\frac{80}{9} \\ -\frac{4}{9} & \frac{20}{9} \end{pmatrix}$

$$E(B, c_1) = L(c_1 I_2 - B) = \mathbb{R}\begin{pmatrix} \frac{80}{9} \\ \frac{16}{9} \end{pmatrix} = \mathbb{R}\begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

<u>Eigenwert $c_2 = -1$:</u> $c_2 I_2 - B = \begin{pmatrix} -\frac{20}{9} & -\frac{80}{9} \\ -\frac{4}{9} & -\frac{16}{9} \end{pmatrix}$

$$E(B, c_2) = L(c_2 I_2 - B) = \mathbb{R}\begin{pmatrix} -\frac{80}{9} \\ \frac{20}{9} \end{pmatrix} = \mathbb{R}\begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

Somit gilt: $T = \begin{pmatrix} 5 & -4 \\ 1 & 1 \end{pmatrix}$, und nach Satz 219

ist $T^{-1} B T = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$.

63) $A = \begin{pmatrix} \frac{1}{18} & \frac{20}{9} \\ \frac{1}{9} & -\frac{1}{18} \end{pmatrix}$, $\det(\lambda I_2 - A) =$

$$= \det \begin{pmatrix} \lambda - \frac{1}{18} & -\frac{20}{9} \\ -\frac{1}{9} & \lambda + \frac{1}{18} \end{pmatrix} = \lambda^2 - \frac{1}{18^2} - \frac{20}{9^2} = 0$$

$$\Longleftrightarrow \lambda^2 = \frac{1+80}{4 \cdot 81} = \frac{1}{4}. \quad \lambda_{1,2} = \pm \frac{1}{2}.$$

<u>Eigenwert $\lambda_1 = \frac{1}{2}$:</u> $\frac{1}{2} I_2 - A = \begin{pmatrix} \frac{8}{18} & -\frac{20}{9} \\ -\frac{1}{9} & \frac{5}{9} \end{pmatrix}$

$$\Rightarrow E(A, \tfrac{1}{2}) = L(\tfrac{1}{2} I_2 - A, 0) = \mathbb{R} \begin{pmatrix} \frac{20}{9} \\ \frac{4}{9} \end{pmatrix} = \mathbb{R} \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

<u>Eigenwert $\lambda_2 = -\frac{1}{2}$:</u> $-\frac{1}{2} I_2 - A = \begin{pmatrix} -\frac{5}{9} & -\frac{20}{9} \\ -\frac{1}{9} & -\frac{4}{9} \end{pmatrix}$

$$\Rightarrow E(A, -\tfrac{1}{2}) = \mathbb{R} \begin{pmatrix} -\frac{20}{9} \\ \frac{5}{9} \end{pmatrix} = \mathbb{R} \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

$$S = \begin{pmatrix} 5 & -4 \\ 1 & 1 \end{pmatrix} \quad S^{-1} A S = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \Rightarrow$$

$$A = S \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} S^{-1} \text{ und } A^n = S \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}^n S^{-1}$$

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}^{10000} = \begin{pmatrix} \frac{1}{2^{10000}} & 0 \\ 0 & \frac{1}{2^{10000}} \end{pmatrix} =$$

$$= \frac{1}{2^{10000}} I_2 \Rightarrow A^{10000} =$$

$$= S \frac{1}{2^{10000}} I_2 S^{-1} = \frac{1}{2^{10000}} I_2 \approx$$

$$5{,}01 \cdot 10^{-3011} \cdot I_2$$