University of Innsbruck
Department of Mathematics

Bachelor Thesis

Topics in non-linear differential equations

Author:
Lukas Einkemmer

Advisor:
Dr. Norbert Ortner

Contents

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1 Preliminaries

1.1 The difficulty of solving non-linear differential equations

Consider for a moment a system of n first-order *linear* differential equations. If we specify (arbitrarily) n initial values, we get a unique solution to our differential equations (assuming certain regularity conditions). However, what makes a *linear* differential equation unique is that once we have obtained any n solutions $x_1(t), \ldots, x_n(t)$ (which must be linearly independent), we can use the fact that any linear combination is a solution as well, i.e. we have only to determine n constants to fully solve the problem for *every* initial condition.

Not so for *non-linear* differential equations. Potentially, we have (qualitatively) very different solutions for different initial conditions. But even if we could, in theory, determine this solution for any given initial condition (by elementary functions, Taylor series, ...), in practice this proves to be a highly formidable task.

Therefore, in the first part of this bachelor thesis we will develop methods that allow us to find approximate solutions to non-linear differential equations which differ only slightly from solvable (that usually means linear) differential equations.

In the second part, our focus will shift to a qualitative assessment of non-linear differential equations. It will be shown that non-linear differential equations provide a range of interesting phenomenons (including deterministic chaos).

1.2 Notation & Prerequisites

The solution of the differential equations considered in this bachelor thesis can be expressed as a function $x : I \to \mathcal{D} \subset \mathbb{R}^n$ with $I = [0, b]$ or $I = [0, b)$, where $b = \infty$ is a possibility in the second case.

We call a differential equation autonomous, if it doesn't depend on time excplicitely, i.e.

Definition 1.1. (autonomous system). A system $\dot{x} = f(x, t)$ of differential equations for $x \in \mathcal{C}^1(I, \mathbb{R}^n)$ and a given function $f \in \mathcal{C}^0(\mathbb{R}^n \times I, \mathbb{R}^n)$ is called autonomous if f doesn't depend on t, i.e.

$$\dot{x} = f(x), \qquad x \in \mathcal{C}^1(I, \mathbb{R}^n), f \in \mathcal{C}^0(\mathbb{R}^n, \mathbb{R}^n)$$

This bachelor thesis continuous to develop methods in the field of non-linear differential equations that go beyond what is usually taught in most introductory courses (e.g. stability theory, Lyapunov functions, Poincaré maps). However, the material presented here doesn't assume familiarity with these topics.

2 Perturbation

2.1 Asymptotic expansion

In mathematics we often use a series expansion to represent non-elementary functions. It is assumed that the series converges to the desired function. In practice, the series is then truncated after a finite number of terms to gain a sufficient approximation. If the series, however, is divergent we call it a *formal* series and are usually forced to use other methods to evaluate the function.

However, in certain cases we can use the truncated (but formal) series and still regain a good approximation. To illustrate this we consider Stirling's approximation to $\log n!$ (see [8, p. 183]).

$$S_m(n) = \left(n + \frac{1}{2}\right) \log(n+1) - n - 1 + \frac{1}{2} \log 2\pi + \sum_{k=1}^{m} \frac{B_{2k}}{2k(2k-1)(n+1)^{2k-1}} = \log(n!) + \mathcal{O}\left(\frac{1}{n^{2m+1}}\right)$$

Let us now calculate the error for some values of n and m.

$$
\begin{aligned}
|\log(10!) - S_2(10)| &\approx 4.9 \cdot 10^{-9} \\
|\log(10!) - S_5(10)| &\approx 6.5 \cdot 10^{-15} \\
|\log(1000!) - S_2(1000)| &\approx 7.9 \cdot 10^{-19} \\
|\log(1000!) - S_5(1000)| &\approx 1.9 \cdot 10^{-36}
\end{aligned}
$$

Note that as n gets larger the approximation improves. This is to be expected since $n \to \infty$ is the limit in which the approximation is exact (see [8, p. 183]). It also seems as if the approximation improves if we include more terms in our truncated series (i.e. m gets larger). This, however, is wrong. Consider

$$
\lim_{k\to\infty} \left| \frac{B_{2k}}{2k(2k-1)(n+1)^{2k-1}} \right| \geq \lim_{k\to\infty} \left| \frac{(2k)!}{k(2k-1)(n+1)^{2k-1}(2\pi)^{2k}} \right| \to \infty
$$

where we used that $|B_{2k}| \geq \frac{2(2k)!}{(2\pi)^{2k}}$ (see [12, p. 361]). Now we see that the series diverges (i.e. is only formal). Nevertheless, the truncated series provides a good approximation to $\log(n!)$.

Stirling's approximation (in logarithmic form) illustrates the idea of *asymptotic expansion*: Even though we have a divergent series at hand, the truncated series is (under certain conditions) an excellent approximation.

2.2 The regular perturbation method [1]

The treatment of perturbative methods in this bachelor thesis is mainly based upon [3, Chap. 3-4] and [5, Chap. 4].

We consider differential equations of the form

$$
\dot{x} = f(x) + \epsilon g(x, t, \epsilon), \qquad x : I \to \mathcal{D} \subset \mathbb{R}^n \tag{1}
$$

and require $f, g \in \mathcal{C}^r$, with $r \in \mathbb{N}$, $r \geq 1$. Furthermore, $\dot{x} = f(x)$ (the **unperturbed** problem) is exactly solvable and ϵ is a small parameter. Our goal is to asymptotically expand the solution $x(t)$ as (see [3, p. 55])

$$
x(t) = x^{(0)}(t) + \epsilon x^{(1)}(t) + \epsilon^2 x^{(2)}(t) + \cdots + \epsilon^{r-1} x^{(r-1)}(t) + \epsilon^r R_r(t, \epsilon) \tag{2}
$$

In general we proceed by feeding equation 2 into 1. If equation 1 is an IVP with $x(0) = x_0$, we further impose the following initial conditions on our asymptotic expansion

$$
x^{(0)}(0) = x_0 \tag{3}
$$

$$
x^{(j)}(0) = 0, \qquad \forall j \geq 1 \tag{4}
$$

Now let us illustrates this method by solving a simple one dimensional example.

Example 2.1. We consider the equation

$$
\dot{x} = -x + \epsilon g(x)
$$

Now we proceed by feeding our ansatz (up to order 2) into this equation

$$
\dot{x}^{(0)} + \epsilon \dot{x}^{(1)} + \epsilon^2 \dot{x}^{(2)} = -x^{(0)} - \epsilon x^{(1)} - \epsilon^2 x^{(2)} + \epsilon g\left(x^{(0)} + \epsilon x^{(1)} + \epsilon^2 x^{(2)}\right)
$$

Expanding g at $x^{(0)}$ we get

$$
\epsilon g\left(x^{(0)} + \epsilon x^{(1)} + \epsilon^2 x^{(2)}\right) = \epsilon g(x^{(0)}) + g'(x^{(0)}) \epsilon^2 x^{(1)} + \mathcal{O}(\epsilon^3)
$$

[1] In most textbooks this is referred to as *regular perturbation theory*; however, this bachelor thesis emphasizes methods and doesn't develop a theory in the formal sense.

Then

$$\dot{x}^{(0)} + \epsilon\dot{x}^{(1)} + \epsilon^2\dot{x}^{(2)} = -x^{(0)} + \epsilon\left(-x^{(1)} + g(x^{(0)})\right) + \epsilon^2\left(-x^{(2)} + g'(x^{(0)})x^{(1)}\right) + \mathcal{O}(\epsilon^3)$$

Equating orders of ϵ

$$\begin{aligned}\dot{x}^{(0)} &= -x^{(0)}\\ \dot{x}^{(1)} &= -x^{(1)} + g(x^{(0)})\\ \dot{x}^{(2)} &= -x^{(2)} + g'(x^{(0)})x^{(1)}\end{aligned}$$

Noting that $x^{(0)}(t) = x_0\mathrm{e}^{-t}$ with $x(0) = x_0$, we solve the next equation by variation of parameters (with initial condition $x^{(1)}(0) = 0$)

$$x^{(1)}(t) = \mathrm{e}^{-t}\int_0^t \mathrm{e}^s g(x_0\mathrm{e}^{-s})\,\mathrm{d}s$$

Then equivalently for $x^{(2)}$ we get

$$x^{(2)}(t) = \mathrm{e}^{-t}\int_0^t \mathrm{e}^s g'(x_0\mathrm{e}^{-s})x^{(1)}(s)\,\mathrm{d}s$$

We now consider the particular case where $g(x) = x^2$. Then

$$\begin{aligned}x^{(1)}(t) &= x_0^2\mathrm{e}^{-t}\int_0^t \mathrm{e}^{-s}\,\mathrm{d}s = x_0^2\mathrm{e}^{-t}\left(1-\mathrm{e}^{-t}\right)\\ x^{(2)}(t) &= 2x_0^3\mathrm{e}^{-t}\int_0^t \mathrm{e}^{-s}\left(1-\mathrm{e}^{-s}\right)\,\mathrm{d}s = x_0^3\mathrm{e}^{-t}\left(1-2\mathrm{e}^{-t}+\mathrm{e}^{-2t}\right)\end{aligned}$$

Finally our expansion reads

$$x(t) = x_0\mathrm{e}^{-t} + \epsilon x_0^2\mathrm{e}^{-t}\left(1-\mathrm{e}^{-t}\right) + \epsilon^2 x_0^3\mathrm{e}^{-t}\left(1-2\mathrm{e}^{-t}+\mathrm{e}^{-2t}\right) + \mathcal{O}(\epsilon^3)$$

Note that in this case we can calculate the exact solution (by the method of separation of an ODE):

$$x(t) = \frac{1}{\epsilon - \mathrm{e}^t(\epsilon - 1/x_0)}$$

Usually, having computed the exact solution would render finding an approximated solution, by the method of perturbation, futile. However, in this case it gives us the possibility to investigate the error we made in our approximation more closely. The reader is advised, however, that perturbative methods are only of practical value if the differential equation can't be solved exactly and that therefore most examples in this bachelor thesis are chosen to illustrate the method at hand (as opposed to illustrating a concrete application).

Figure 1 gives an idea of the magnitude of the absolute error we made in this approximation (for different ϵ).

In the previous example we computed an asymptotic expansion of order 2 and compared it with the exact solution. Now we are interested in an error estimate. To accomplish this we first need one version of Gronwall's inequality (see [3, p. 55]).

Lemma 2.1. ***(Gronwall's inequality)*** *If $x, \alpha, \beta \in \mathcal{C}^0(I, \mathbb{R})$ with $\beta(t) \geq 0$, $\forall t \in I$, and*

$$x(t) \leq \alpha(t) + \int_0^t \beta(s)x(s)\,\mathrm{d}s \qquad \forall t \in I \tag{5}$$

Then

$$x(t) \leq \alpha(t) + \int_0^t \beta(s)\alpha(s)\mathrm{e}^{\int_s^t \beta(u)\,\mathrm{d}u}\,\mathrm{d}s \qquad \forall t \in I \tag{6}$$

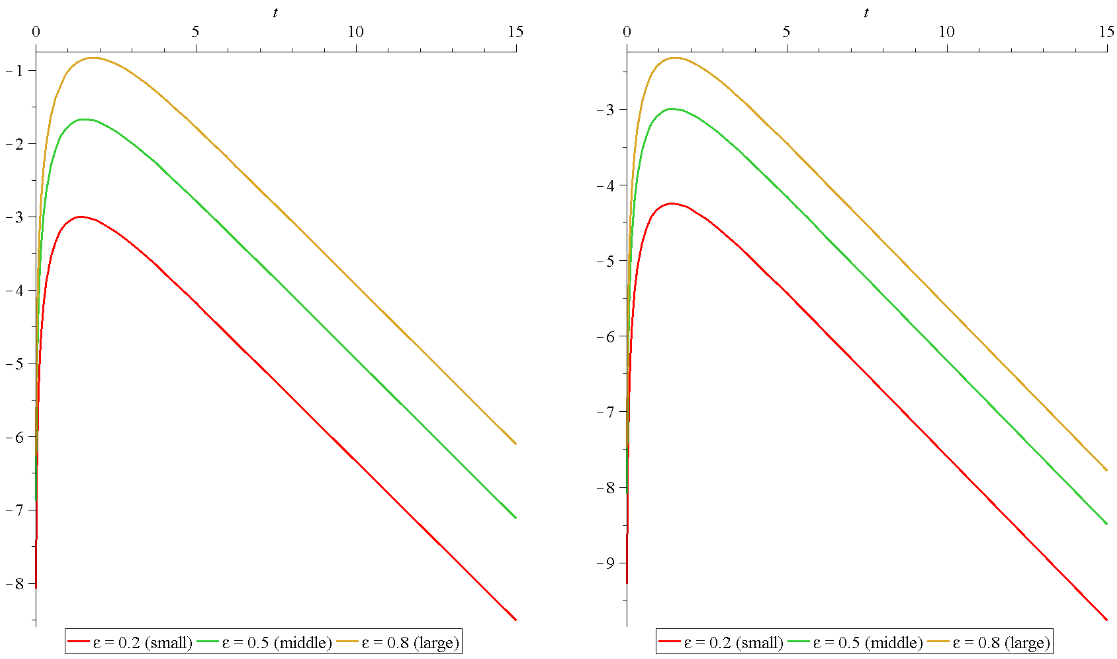

Figure 1: Logarithmic (base 10) plot of the absolute error for $x_0 = 1$ (left) and $x_0 = 0.5$ (right)

Proof. Define

$$R(t) = \int_0^t \beta(s)x(s)\,\mathrm{d}s$$

Since $\beta(t) \geq 0$

$$\dot{R}(t) = \beta(t)x(t) \leq \beta(t)\alpha(t) + \beta(t)R(t)$$

Let $B(s) = \int_0^s \beta(u)\,\mathrm{d}u$. Then

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\mathrm{e}^{-B(s)}R(s)\right) = \left(R'(s) - \beta(s)R(s)\right)\mathrm{e}^{-B(s)} \leq \beta(s)\alpha(s)\mathrm{e}^{-B(s)}$$

and by integrating from 0 to t

$$\mathrm{e}^{-B(t)}R(t) \leq \int_0^t \beta(s)\alpha(s)\mathrm{e}^{-B(s)}\,\mathrm{d}s$$

multiplying by $\mathrm{e}^{-B(t)}$ and inserting the result into equation 5

$$\begin{aligned} x(t) &\leq \alpha(t) + \int_0^t \beta(s)\alpha(s)\mathrm{e}^{\int_0^t \beta(u)\,\mathrm{d}u - \int_0^s \beta(u)\,\mathrm{d}u} \\ &= \alpha(t) + \int_0^t \beta(s)\alpha(s)\mathrm{e}^{\int_s^t \beta(u)\,\mathrm{d}u} \end{aligned}$$

we get equation 6 as desired. □

Before we proceed to state and proof an estimate of the absolute error made in our approximation, a note of caution is in order. In the following discussion $\|g(x,t,\epsilon)\|$ always refers to a norm of $\mathbb{R}^n$ evaluated at a single point (in the domain of g). We don't need to introduce a norm on a function (e.g. the uniform norm).

Since all norms in $\mathbb{R}^n$ are equivalent, it makes no difference which norm we choose (The Euclidean norm $\|\cdot\|_2$ is usually the canonical choice).

Theorem 2.1. *Assume f is Lipschitz-continuous with Lipschitz constant K and $\|g(x,t,\epsilon)\|$ is uniformly bounded by M on $\mathcal{D}\times I\times[0,\epsilon_0]$. If $x(t)$, $x^{(0)}(t)$ are defined as in equation 2 and $x(0)=x^{(0)}(0)$, then*

$$\left\|x(t)-x^{(0)}(t)\right\| \leq \frac{\epsilon M}{K}\left(\mathrm{e}^{Kt}-1\right), \qquad \forall t\in I \tag{7}$$

Proof. We consider $y=x-x^{(0)}$. Then

$$\begin{aligned}\dot{y}=\dot{x}-\dot{x}^{(0)} &= f(x)-f(x^{(0)})+\epsilon g(x,t,\epsilon)\\ &= f(y+x^{(0)})-f(x^{(0)})+\epsilon g(x,t,\epsilon)\end{aligned}$$

Considering the integral of this equation

$$y(t)=\int_0^t f(x^{(0)}(s)+y(s))-f(x^{(0)}(s))\,\mathrm{d}s+\epsilon\int_0^t g(x(s),s,\epsilon)\,\mathrm{d}s$$

Then

$$\begin{aligned}\|y(t)\| &\leq \int_0^t \left\|f(x^{(0)}(s)+y(s))-f(x^{(0)}(s))\right\|\,\mathrm{d}s+\epsilon\int_0^t \|g(x(s),s,\epsilon)\|\,\mathrm{d}s\\ &\leq \int_0^t K\,\|y(s)\|\,\mathrm{d}s+\epsilon Mt\end{aligned}$$

since (by the Lipschitz condition) $\left\|f(x^{(0)}(s)+y(s))-f(x^{(0)}(s))\right\| \leq K\,\|y(s)\|$. Then we apply Gronwall's inequality and get

$$\begin{aligned}\|y(t)\| &\leq \epsilon Mt+\epsilon MK\int_0^t s\mathrm{e}^{\int_s^t K\,\mathrm{d}u}\,\mathrm{d}s\\ &= \frac{\epsilon M}{K}\left(\mathrm{e}^{Kt}-1\right)\end{aligned}$$

as desired. □

Example 2.2. The Van der Pol oscillator is described by (see [5, p. 67])

$$\ddot{x}+\alpha(1-x^2)\dot{x}+x=\beta p(t) \tag{8}$$

or equivalently

$$\ddot{x}=-\frac{\mathrm{d}}{\mathrm{d}t}\left[\alpha\left(x-\frac{x^3}{3}\right)\right]-x+\beta p(t)$$

we rewrite it as a two-dimensional first order equation with (by definition[2]) $y=\dot{x}+\alpha\left(x-\frac{x^3}{3}\right)$

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}=\begin{bmatrix}y+\beta\Theta(x)\\ -x+\beta p(t)\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}+\epsilon\begin{bmatrix}\Theta(x)\\ p(t)\end{bmatrix} \tag{9}$$

with $\Theta(x)=-\frac{\alpha}{\beta}\left(x-\frac{x^3}{3}\right)$, $\epsilon=\beta$ and initial conditions $x(0)=x_0$ and $y(0)=y_0$. Therefore, to carry out the regular perturbation method we must require that β is small and that $\alpha/\beta\approx 1$. Feeding our ansatz (equation 2) into equation 9 (up to order ϵ) we get

$$\begin{bmatrix}\dot{x}^{(0)}\\ \dot{y}^{(0)}\end{bmatrix}+\epsilon\begin{bmatrix}\dot{x}^{(1)}\\ \dot{y}^{(1)}\end{bmatrix}=\begin{bmatrix}y^{(0)}\\ -x^{(1)}\end{bmatrix}+\epsilon\begin{bmatrix}y^{(1)}+\Theta(x^{(0)}+\epsilon x^{(1)})\\ -x^{(1)}+p(t)\end{bmatrix} \tag{10}$$

Now we can easily solve the linear system describing $x^{(0)}$ and $y^{(0)}$

$$\begin{bmatrix}x^{(0)}\\ y^{(0)}\end{bmatrix}=\begin{bmatrix}y_0\sin t+x_0\sin t\\ y_0\cos t-x_0\sin t\end{bmatrix}$$

[2] We avoid the canonical choice of $y=\dot{x}$ to find a single ϵ valid for the entire system.

Then plugging this solution into the terms of order ϵ in equation 10 we get

$$\begin{bmatrix} \dot{x}^{(1)} \\ \dot{y}^{(1)} \end{bmatrix} = \begin{bmatrix} y^{(1)} + \Theta\left(x^{(0)} + \epsilon x^{(1)}\right) \\ -x^{(1)} + p(t) \end{bmatrix} = \begin{bmatrix} y^{(1)} + \Theta(x^{(0)}) + \mathcal{O}(\epsilon) \\ -x^{(1)} + p(t) \end{bmatrix}$$

where $x^{(0)}$ has already been determined and we can ignore terms of order ϵ. Now we have an inhomogeneous linear differential equation given by

$$\begin{bmatrix} \dot{x}^{(1)} \\ \dot{y}^{(1)} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x^{(1)} \\ y^{(1)} \end{bmatrix} + \begin{bmatrix} \Theta \\ p \end{bmatrix}, \qquad \begin{matrix} x^{(1)}(0) = 0 \\ y^{(1)}(0) = 0 \end{matrix}$$

where we write $\Theta = \Theta(x^{(0)}(t))$ and $p = p(t)$ since both functions only depend on t now and no confusion can arise. To solve this system we diagonalize the matrix (call it A) and get

$$T = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}, T^{-1} = \frac{1}{2}\begin{bmatrix} -i & 1 \\ -i & 1 \end{bmatrix}, \qquad T^{-1}AT = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

and

$$T^{-1}\begin{bmatrix} \Theta \\ p \end{bmatrix} = \frac{1}{2}\begin{bmatrix} i\Theta + p \\ -i\Theta + p \end{bmatrix}$$

Now we need to solve

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\hat{x}^{(1)} &= i\hat{x}^{(1)} + \frac{1}{2}\left(i\Theta + p\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\hat{y}^{(1)} &= -i\hat{y}^{(1)} + \frac{1}{2}\left(-i\Theta + p\right) \end{aligned}$$

Which are two uncoupled linear differential equations. Thus, by the method of variation of parameters

$$\begin{bmatrix} \hat{x}^{(1)} \\ \hat{y}^{(1)} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \mathrm{e}^{it}\int_0^t \mathrm{e}^{-is}\left(i\Theta(x^{(0)}(s)) + p(s)\right)\,\mathrm{d}s \\ \mathrm{e}^{-it}\int_0^t \mathrm{e}^{is}\left(-i\Theta(x^{(0)}(s)) + p(s)\right)\,\mathrm{d}s \end{bmatrix}$$

reversing the coordinate transformation we get

$$\begin{bmatrix} x^{(1)} \\ y^{(1)} \end{bmatrix} = T\begin{bmatrix} \hat{x}^{(1)} \\ \hat{y}^{(1)} \end{bmatrix} = \begin{bmatrix} \frac{1}{i}\left(\hat{x}^{(1)} - \hat{y}^{(1)}\right) \\ \hat{x}^{(1)} + \hat{y}^{(1)} \end{bmatrix}$$

Thus, we have computed our expansion up to order 1. Now let us apply theorem 2.1 to get an estimate of the absolute error. To simplify this calculation we assume that α=β; furthermore, we will use the Euclidean norm explicitly. First, note that since $f(x, y) = (y, -x)^T$ is a linear function its Lipschitz constant is equal to the operator norm of f. By a well-known theorem from linear algebra we therefore get the Lipschitz constant as the square-root of the largest eigenvalue of A^TA where A is a matrix associated with the linear map f.

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, A^TA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \implies K = \sqrt{\lambda_{\max}} = 1$$

We now need to find a bound of

$$\|g(x, t, \epsilon)\|_2 = \left\|\begin{bmatrix} \Theta(x) \\ p(t) \end{bmatrix}\right\|_2 = \sqrt{\Theta(x)^2 + p(t)^2}$$

Since $|\Theta(x)|$ is unbounded if $x \to \infty$, we limit our discussion to $x(t) \in (-1, 1)$ (this effectively means that our error estimate is only valid for solutions that stay close to 0); we have $\sup\{|\Theta(x)|\,;\, x \in (-1, 1)\} = \frac{2}{3}$. Additionally assume $\sup\{|p(t)|\,;\, t \in [0, \infty)\} = P$. Then

$$\|g(x, t, \epsilon)\|_2 \le \sqrt{4/9 + P^2}$$

Thus, our estimate reads

$$\left\| \begin{bmatrix} x(t) - x^{(0)}(t) \\ y(t) - y^{(0)}(t) \end{bmatrix} \right\|_2 \leq \beta \sqrt{4/9 + P^2} \left(e^t - 1 \right)$$

2.3 Averaging

We consider differential equations of the form

$$\dot{x} = \epsilon g(x, t, \epsilon), \qquad x : I \to \mathcal{D} \subset \mathbb{R}^n \tag{11}$$

and require $g \in \mathcal{C}^r$, with $r \in \mathbb{N}$, $r \geq 2$. Additionally, g is T-periodic in t, i.e. $g(x, t+T, \epsilon) = g(x, t, \epsilon)$. By theorem 2.1 the error of the trivial approximation $x^{(0)}(t) = x(0)$ is bounded by (this is a special case of $K = 0$ in equation 1)

$$\|x(t) - x(0)\| \leq \lim_{K \to 0} \left[\frac{\epsilon M}{K} \left(\mathrm{e}^{Kt} - 1 \right) \right] = \epsilon M t$$

However, we can improve on this estimate further by averaging out the periodic influence of g (see [5, p. 168]).

Definition 2.1. The averaged system associated with equation 11 is the autonomous equation

$$\dot{y}^{(0)} = \epsilon \bar{g}(y^{(0)}), \qquad \bar{g}(x) = \frac{1}{T} \int_0^T g(x, t, 0) \, \mathrm{d}t \tag{12}$$

Theorem 2.2. *There exists a change of coordinates $x = y + \epsilon w(y, t)$, with $w \in \mathcal{C}^r$ and w T-periodic in t, which transforms equation 11 into*

$$\dot{y} = \epsilon \bar{g}(y) + \epsilon^2 g_1(y, t, \epsilon) \tag{13}$$

where g_1 is T-periodic in t as well (see [5, p. 168] or [3, p. 58]).

Proof. First, we write g as

$$g(x, t, \epsilon) = \bar{g}(x) + g^{(0)}(x, t, \epsilon)$$

with $\overline{g^{(0)}}(x) = 0$. Inserting our coordinate transform $x = y + \epsilon w(y, t)$ into equation 11 we get

$$\dot{x} = \left(I + \epsilon \frac{\partial w}{\partial y} \right) \dot{y} + \epsilon \frac{\partial w}{\partial t} = \epsilon \bar{g}(y + \epsilon w) + \epsilon g^{(0)}(y + \epsilon w, t, \epsilon)$$

Expanding y we get

$$\dot{x} = \dot{y}^{(0)} + \epsilon \left(\dot{y}^{(1)} + \frac{\partial w}{\partial y} \dot{y}^{(0)} \right) + \epsilon^2 \left(\dot{y}^{(2)} + \frac{\partial w}{\partial y} \dot{y}^{(1)} \right) + \epsilon \frac{\partial w}{\partial t} + \mathcal{O}(\epsilon^3) = \epsilon \bar{g}(y + \epsilon w) + \epsilon g^{(0)}(y + \epsilon w, t, \epsilon)$$

Noting that $y^{(0)} = 0$ (by comparing coefficients in the above equation) we get

$$\epsilon \dot{y}^{(1)} = \epsilon \left[\bar{g}(y + \epsilon w) + g^{(0)}(y + \epsilon w, t, \epsilon) - \frac{\partial w}{\partial t} \right]$$

Now we expand $\bar{g}$ and $g^{(0)}$ in a Taylor series around y. Then

$$\begin{aligned} \bar{g}(y + \epsilon w) &= \bar{g}(y) + \bar{g}'(y) \epsilon w + \mathcal{O}(\epsilon^2) \\ g^{(0)}(y + \epsilon w, t, \epsilon) &= g^{(0)}(y, t, \epsilon) + \frac{\partial g^{(0)}(y, t, \epsilon)}{\partial y} \epsilon w + \mathcal{O}(\epsilon^2) \\ &= g^{(0)}(y, t, 0) + \frac{\partial g^{(0)}(y, t, \epsilon)}{\partial y} \Big|_{\epsilon = 0} \epsilon w + \mathcal{O}(\epsilon^2) \end{aligned}$$

In the last step we used the fact that $g^{(0)}$ can be expanded (at $\epsilon = 0$) as $g^{(0)}(y,t,\epsilon) = g^{(0)}(y,t,0) + \mathcal{O}(\epsilon)$ Inserting this into our equation for $\dot{y}$ we get

$$\dot{y} = \epsilon \left[\bar{g}(y) + g^{(0)}(y,t,0) - \frac{\partial w}{\partial t} \right] + \mathcal{O}(\epsilon^2)$$

Now choosing

$$w(y,t) = \int_0^t g^{(0)}(y,s,0)\,\mathrm{d}s$$

yields the desired result since

$$\dot{y} = \epsilon \bar{g}(y) + \mathcal{O}(\epsilon^2)$$

Furthermore, w is T-periodic since

$$\begin{aligned} w(y,t+T) &= \int_0^T g^{(0)}(y,s,0)\,\mathrm{d}s + \int_T^{T+t} g^{(0)}(y,s,0)\,\mathrm{d}s \\ &= \int_T^{t+T} g^{(0)}(y,s,0)\,\mathrm{d}s \\ &= \int_0^t g^{(0)}(y,s,0)\,\mathrm{d}s \end{aligned}$$

Additionally, we compute g_1, using $y^{(1)}$.

$$g_1(y,t,\epsilon) = \left[\bar{g}'(y) + \frac{\partial g^{(0)}(y,t,0)}{\partial y}\Big|_{\epsilon=0} \right] w - \frac{\partial w}{\partial y} \bar{g}$$

which is T-periodic in t since all the involved functions are. □

We now continue our investigation of the error introduced by solving the averaged system (One should note that in general we don't require $y^{(0)}(0) = y(0)$, since often it is not possible to compute $y(0)$ by solving the equation $x(0) = y(0) + \epsilon w(y(0), 0)$).

Theorem 2.3. *If $y(t)$ and $y^{(0)}(t)$ are solutions of equations 11 and 12 respectively, $\bar{g}$ is Lipschitz continuous with Lipschitz constant K and M is a bound for $\|g_1(y,t,\epsilon)\|$, then*

$$\left\| y(t) - y^{(0)}(t) \right\| \leq \left\| y(0) - y^{(0)}(0) \right\| \mathrm{e}^{\epsilon K t} + \frac{\epsilon M}{K}\left(\mathrm{e}^{\epsilon K t} - 1\right) \tag{14}$$

Proof. Define $z(t) = y(t) - y^{(0)}(t)$, then

$$\dot{z} = \epsilon \left[\bar{g}(y) - \bar{g}(y^{(0)}) \right] + \epsilon^2 g_1(y,t,\epsilon)$$

Now we proceed as in the proof of theorem 2.1. Since

$$z(t) = z(0) + \int_0^t \epsilon \left[\bar{g}(y) - \bar{g}(y^{(0)}) \right] + \epsilon^2 g_1(y,s,\epsilon)\,\mathrm{d}s$$

we have

$$\begin{aligned} \|z(t)\| &\leq \|z(0)\| + \int_0^t \left\| \epsilon \left[\bar{g}(y) - \bar{g}(y^{(0)}) \right] \right\| + \left\| \epsilon^2 g_1(y,s,\epsilon) \right\| \,\mathrm{d}s \\ &\leq \|z(0)\| + \int_0^t \epsilon K \|z(s)\| \,\mathrm{d}s + \epsilon^2 M t \end{aligned}$$

and by Gronwall's inequality

$$\|z(t)\| \leq \|z(0)\| \,\mathrm{e}^{\epsilon K t} + \frac{\epsilon M}{K}\left(\mathrm{e}^{\epsilon K t} - 1\right)$$

as desired. □

One should note that even though this estimate is not linear but exponential, it is a better estimate if t is small (due to the small factor ϵK in the exponential).

Example 2.3. To illustrate the method of averaging, consider the differential equation ([5, p. 171])

$$\dot{x} = \epsilon x \sin^2 t$$

Then the averaged function is given by

$$\bar{g} = \frac{1}{2\pi} \int_0^{2\pi} x \sin^2 t \, \mathrm{d}s = \frac{x}{2}$$

with $g^{(0)}(x,t) = x\left(\sin^2 t - \frac{1}{2}\right) = -\frac{1}{2} x \cos 2t$. According to the proof of theorem 2.2, we get

$$w(y,t) = \int_0^t -\frac{1}{2} y \cos 2s \, \mathrm{d}s = -\frac{1}{4} y \sin 2t$$

Then

$$\begin{aligned}\dot{y} &= \epsilon \frac{1}{2} y - \epsilon^2 \frac{1}{4} y \sin 2t \left[\frac{1}{2} - \frac{1}{2}\cos 2t\right] + \epsilon^2 \frac{1}{8} y \sin 2t \\ &= \epsilon \frac{1}{2} y + \epsilon^2 \frac{1}{8} y \sin 2t \cos 2t \\ &= \epsilon \frac{1}{2} y + \epsilon^2 \frac{1}{16} y \sin 4t\end{aligned}$$

If we are just interested in the averaged equation, we don't need to calculate g_1. Either way, the averaged system

$$\dot{y} = \epsilon \frac{1}{2} y$$

admits the solution

$$y(t) = y_0 \mathrm{e}^{\frac{\epsilon}{2} t}$$

which completes our example.

2.4 Iterative methods

The averaging method allows us to obtain a better approximation (compared to the perturbation method introduced in section 2.2) that is valid for a longer time interval. We now apply this method iteratively to obtain a still better approximation.

Consider the equation

$$\dot{y} = \epsilon \bar{g}(y) + \epsilon^2 g_1(y,t,\epsilon)$$

as in the previous section. Then proceeding in the same way as in theorem 2.2 but using the coordinate transformation $y = y_2 + \epsilon^2 w_2(y_2,t)$, we get (since all the other assumptions of theorem 2.2 are still valid)

$$\dot{y}_2 = \epsilon \bar{g}(y_2) + \epsilon^2 \bar{g}_1(y_2) + \epsilon^3 g_2(y_2,t,\epsilon)$$

The associated averaged system is

$$\dot{y}^{(0)} = \epsilon \bar{g}(y_2) + \epsilon^2 \bar{g}_1(y_2)$$

Now let K_2 and M_2 denote the Lipschitz constant for $\bar{g} + \epsilon \bar{g}_1$ and a bound of $\|g_2(y_2,t,\epsilon)\|$ respectively, we proceed as in theorem 2.3 and get

$$\left\| y_2(t) - y_2^{(0)}(t) \right\| \leq \left\| y_2(0) - y_2^{(0)}(0) \right\| \mathrm{e}^{\epsilon K_2 t} + \frac{\epsilon^2 M_2}{K_2} \mathrm{e}^{\epsilon K_2 t}$$

We can repeat this procedure iteratively. Note, however, that in computing g_1 (see 2.2) we used the first derivative of g. Therefore, we can continue this iteration only if g_1 is differentiable as well (i.e. for $g \in \mathcal{C}^r$ we can compute at most r iterations).

2.5 The singular perturbation method

We consider differential equations of the form

$$\begin{aligned} \epsilon \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned} \tag{15}$$

and $x : I \to \mathbb{R}^n$, $y : I \to \mathbb{R}^m$, f and g are of class $\mathcal{C}^r$ for some $r \in \mathbb{N}$, $r \geq 1$.

We call, for obvious reasons, x the fast variable and y the slow variable. In the limit $\epsilon \to 0$ equation 15 doesn't reduce to a system of differential equations, but to a combination of algebraic and differential equations.

$$\begin{aligned} 0 &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned} \tag{16}$$

However, in this case it is not clear how the solution of equation 15 can be expanded into powers of ϵ given the solution of equation 16.

Another approach is to introduce the *fast time* $s = t/\epsilon$, then

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}s} &= f(x, y) \\ \frac{\mathrm{d}y}{\mathrm{d}s} &= \epsilon g(x, y) \end{aligned} \tag{17}$$

In the limit $\epsilon \to 0$, this reduces to (One should note that as far as the fast time is concerned ϵ is taken to be a constant, i.e. the time is scaled by a fixed parameter)

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}s} &= f(x, y) \\ y &= C \end{aligned} \tag{18}$$

for some constant $C \in \mathbb{R}^m$. Thus, y acts as a parameter. Given we can solve this equation (let us call this solution $x^{(0)}$), we can apply the regular perturbation method to equation 17. Note, however, that $s = t/\epsilon$. Inserting this into the result of theorem 2.1 gives us (where x denotes the solution of equation 15)

$$\left\| x(t) - x^{(0)}(t) \right\| \leq \frac{\epsilon M}{K} \left(\mathrm{e}^{Kt/\epsilon} - 1 \right)$$

i.e. the bound is much worse and most likely only acceptable for extremely small t. To alleviate this, we, for large t, return to equation 16. If we are able to solve this system exactly, i.e. we can solve $f(x, y) = 0$ by $x = x^{\star}(y)$ and find a solution to

$$\dot{y} = g(x^{\star}(y), y)$$

then we have an approximation to the solution of equation 15 that is valid for large t.

Now we can, in some cases, try to combine the solution of equation 18 and equation 16 (which are valid for small and large t respectively) to a single solution which is valid for all t. Since we must determine a point at which we *match* both solutions (i.e. both solutions have the same value) this method is referred to as the **method of matched asymptotic expansion** (see [7, p. 28]). We illustrate this method in the next example.

Example 2.4. Consider the following differential equation

$$\epsilon \ddot{x} + \dot{x} + x = 0 \tag{19}$$

with lower boundary condition $x(0) = 0$ and upper boundary condition $x(1) = 1$. Written as a first-order system, we get

$$\begin{aligned} \epsilon \dot{x} &= -x - y \\ \dot{y} &= x \end{aligned} \tag{20}$$

with the same boundary conditions. Now considering $\epsilon \to 0$, we get $x = -y$ and $\dot{y} = x$; thus, $\dot{y} = -y$, which is easily solved to be

$$x(t) = -y(t) = C\mathrm{e}^{-t} \tag{21}$$

Introducing the fast time $s = t/\epsilon$, we get $y = -D$ (some arbitrary constant) and

$$\frac{\mathrm{d}x}{\mathrm{d}s} = -x + D$$

which is readily solved to be

$$x(t) = D + E\mathrm{e}^{-t/\epsilon}$$

Now, as mentioned before, this solution is only a good approximation for small t, so we require it to satisfy the lower boundary condition $x(0) = 0$. Then

$$x(t) = -E + E\mathrm{e}^{-t/\epsilon} \tag{22}$$

On the other hand we require equation 21 to satisfy the upper boundary condition, then

$$x(t) = \mathrm{e}^{1-t} \tag{23}$$

We now choose E such that the two solutions can be combined in a suitable way (this is what is actually referred to as **matching**). In this case we choose $t = \sqrt{\epsilon}$ as the matching point, i.e. we require (Note that we have no general method to determine an optimal matching point)

$$\lim_{\epsilon \to 0} \mathrm{e}^{1-\epsilon} = \lim_{\epsilon \to 0} \left(-E + E\mathrm{e}^{-1/\sqrt{\epsilon}}\right)$$

where E is readily determined to be

$$E = -\mathrm{e}$$

Adding the two functions and subtracting their common value gives us

$$x(t) \approx \mathrm{e}^{1-t} - \mathrm{e}\left(\mathrm{e}^{-t/\epsilon} - 1\right) - \mathrm{e} = \mathrm{e}\left(\mathrm{e}^{-t} - \mathrm{e}^{-t/\epsilon}\right)$$

Similar to example 2.1 we can find an exact solution to equation 19

$$x(t) = \frac{\mathrm{e}^{\frac{\sqrt{1-4\epsilon}-1}{2\epsilon}t} - \mathrm{e}^{\frac{-\sqrt{1-4\epsilon}-1}{2\epsilon}t}}{\mathrm{e}^{\frac{\sqrt{1-4\epsilon}-1}{2\epsilon}} - \mathrm{e}^{\frac{-\sqrt{1-4\epsilon}-1}{2\epsilon}}}$$

and use it to investigate the error we made in our approximation more closely. Figure 2 gives an idea of the qualitative behavior of the solution as well as of the absolute error made in our approximation (It should be noted that as $t \to 0$ and as $t \to \infty$ the error is negligible).

3 Qualitative analysis

3.1 Introduction to dynamical systems

The treatment in this section is mainly based on [14, Chap. 23-25,29,30] and [5, Chap. 5].

Although more general definitions exist (see e.g. [1, p. 328-331]), since we are interested only in dynamical systems with values in $\mathbb{R}^n$, the following definition will suffice.

Definition 3.1. (Dynamical system) If T is a monoid (written additively), then

$$\Phi : T \times \mathbb{R}^n \to \mathbb{R}^n$$

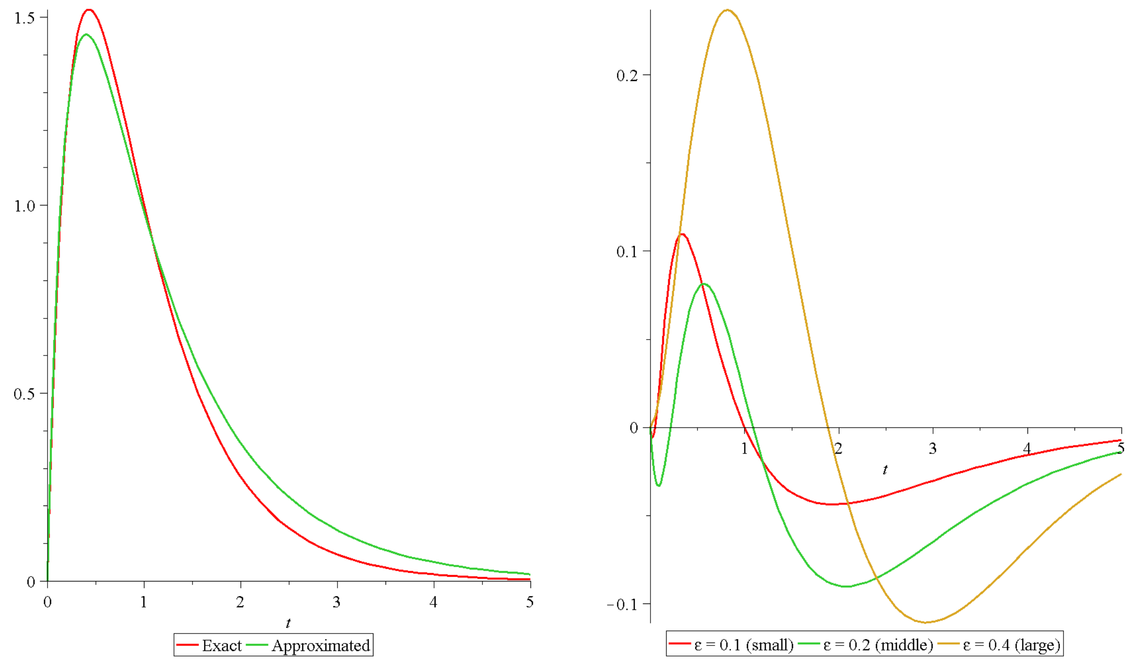

Figure 2: Function graph for the exact and approximated solution (left) and absolute error plot (*not* logarithmic) (right).

is a dynamical system if

$$\Phi(0, x) = x \tag{24}$$

$$\Phi(t_2, \Phi(t_1, x)) = \Phi(t_1 + t_2, x), \qquad \forall t_1, t_2 \in T,\ t_1 + t_2 \in T \tag{25}$$

If we take x as a fixed parameter, we call Φ the **flow** of the dynamical system through x and

$$\gamma_x = \{\Phi(t, x) \,|\, t \in T\}$$

the **orbit** through x.

A dynamical system as defined above is an abstract concept. The two most important models for our purpose are $T \subset \mathbb{R}$ and $T \subset \mathbb{Z}$.

Definition 3.2. (Continuous/Discrete dynamical system)

1. If $T \subset \mathbb{R}$, we call the dynamical system **continuous**.
2. If $T \subset \mathbb{Z}$, we call the dynamical system **discrete**.

We are already familiar with continuous dynamical systems, since those are (for our purpose) given by autonomous differential equations as shown below.

Theorem 3.1. *Every autonomous system which permits a unique solution for any initial condition $x(0) = x_0$ is a dynamical system.*

Proof. Define $\Phi(t, x_0) = x(t)$ where $\dot{x} = f(x)$ and $x(0) = x_0$. Then

$$\Phi(0, x_0) = x(0) = x_0$$

and since, per assumption, $x(t)$ is unique

$$\Phi(t_1 + t_2, x_0) = x(t_1 + t_2)$$

then $\Phi(t_1, x_0) = x(t_1)$, but since the flow through a point is unique, it follows that

$$\Phi(t_2, \Phi(t_1, x_0)) = x(t_1 + t_2)$$

as desired. □

Theorem 3.2. *Every system of first-order differential equations of the form*

$$\dot{x} = f(x,t) \qquad x \in \mathcal{C}^1(I, \mathbb{R}^n),\ f \in \mathcal{C}^0(\mathbb{R}^n \times I, \mathbb{R}^n)$$

in n-variables can be written as an autonomous system in $n+1$ variables.

Proof. We introduce a new variable τ and write

$$\begin{aligned} \dot{x} &= f(x, \tau) \\ \dot{\tau} &= 1 \end{aligned}$$

with initial condition $(x, \tau)(t) = (x_0, 0)$. This is an autonomous system in $n+1$ variables. Now solve for τ which is an uncoupled variable to get $\tau = t$ as desired. □

The reader might be less familiar with discrete dynamical systems. The next example should illustrate the concept.

Example 3.1. The **discrete logistic map** (see [13])

$$x_{n+1} = f(x_n) = r x_n (1 - x_n), \qquad x_0 = a,\ f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$$

is a quadratic recurrence relation. We will show that the logistic map indeed satisfies the requirements of a discrete dynamical system. First, $\Phi(0, a) = x_0 = a$. Second,

$$\begin{aligned} \Phi(n_1 + n_2, a) &= f^{n_1+n_2}(a) \\ \Phi(n_2, \Phi(n_1, a)) &= \Phi(n_2, f^{n_1}(a)) = f^{n_1+n_2}(a) \end{aligned}$$

thus, the logistic map is a discrete dynamical system.

3.2 Lyapunov exponent

We consider an autonomous system, i.e.

$$\dot{x} = f(x) \tag{26}$$

and require $x \in \mathcal{C}^2(I, \mathbb{R}^n),\ f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$.

Suppose we start with the flow through x_0 given by $\Phi(t, x_0)$. Now consider an infinitesimal deviation u from this flow. We will investigate, how the flow $\Phi(t, x_0 + u)$ diverges from $\Phi(t, x_0)$ as $t \to \infty$. Since we consider flows in the vicinity of $\Phi(t, x_0)$, we linearize equation 26 around $\Phi(t, x_0)$

$$\dot{y} = f'(\Phi(t, x_0))y \tag{27}$$

Equation 27 is a system of linear differential equations with the fundamental solution matrix $X(t, x_0)$. Now we define the *coefficient of expansion in the direction u along the flow* $\Phi(t, x_0)$ as

$$\lambda(t, x_0, u) = \frac{\|X(t, x_0)u\|}{\|u\|}$$

Now let us assume that the linearized flow diverges exponentially, i.e.

$$\lambda(t, x_0, u) = \mathrm{e}^{\chi \cdot t}$$

In the limit as $t \to \infty$ the constant χ as given above is called the Lyapunov exponent. Thus, we define

Definition 3.3. (Lyapunov exponent)

$$
\begin{aligned}
\chi(X, x_0, u) &= \limsup_{t\to\infty} \frac{1}{t} \log \lambda(t, x_0, u) \\
\chi(X, x_0, 0) &= -\infty
\end{aligned} \tag{28}
$$

From now on we drop the first two parameters of χ and write $\chi(u) = \chi(X, x_0, u)$ instead. Note that the Lyapunov exponent for a given x_0 depends on u, potentially giving rise to a multitude of different Lyapunov exponents. It will be shown (in the next theorem) that the number of such Lyapunov exponents is at most n.

Lemma 3.1. *$u \in \mathbb{R}^n$ and $c \in \mathbb{R}$ then*

$$
\begin{aligned}
\chi(u + v) &\le \max\{\chi(u), \chi(v)\} \\
\chi(cu) &= \chi(u)
\end{aligned}
$$

and $\{g \in \mathbb{R}^n \mid \chi(g) \le c\}$ is a (vector) subspace of $\mathbb{R}^n$.

Proof. For brevity we will write $X = X(t, x_0)$ in this proof.

1.
$$\chi(cu) = \limsup_{t\to\infty} \frac{1}{t} \frac{\|Xcu\|}{\|cu\|} = \limsup_{t\to\infty} \frac{1}{t} \frac{\|Xu\|}{\|u\|} = \chi(u)$$

2. Note that
$$\limsup_{t\to\infty} \frac{1}{t} \log \frac{\|Xu\|}{\|u\|} = \limsup_{t\to\infty} \left(\frac{\log \|Xu\|}{t} - \frac{\log \|u\|}{t} \right) = \limsup_{t\to\infty} \frac{1}{t} \log \|Xu\|$$
Thus, we need to proof that
$$\limsup_{t\to\infty} \frac{1}{t} \log \|X(u+v)\| \le \max \left\{ \limsup_{t\to\infty} \frac{1}{t} \log \|Xu\|, \limsup_{t\to\infty} \frac{1}{t} \log \|Xv\| \right\}$$
Now, consider w.l.o.g. that $\chi(u) \ge \chi(v)$. Then $\chi(u) - \chi(v) \ge 0$ and
$$\left(\lim_{s\to\infty} \sup_{t\in(s,\infty)} \left\{ \frac{\log \|Xu\|}{t} - \frac{\log \|Xv\|}{t} \right\} \ge 0 \right) \implies \left(\lim_{s\to\infty} \sup_{t\in(s,\infty)} \{\log \|Xu\| - \log \|Xv\|\} \ge 0 \right)$$
Therefore
$$
\begin{aligned}
\limsup \frac{1}{t} \log \|X(u+v)\| &\le \limsup \frac{1}{t} \log (\|Xu\| + \|Xv\|) \\
&= \log \limsup_{t\to\infty} (\|Xu\| + \|Xv\|)^{1/t} \\
&\le \log \limsup_{t\to\infty} (2\|Xu\|)^{1/t} \\
&= \log \limsup_{t\to\infty} \|Xu\|^{1/t} \\
&= \limsup_{t\to\infty} \frac{1}{t} \log \|Xu\|
\end{aligned}
$$
as desired.

3. By the previous results and since $0 \in \{g \in \mathbb{R}^n \mid \chi(g) \le c\}$, $\forall c \in \mathbb{R}$, we have a subspace.

□

Theorem 3.3. ***There are at most n (finite) Lyapunov exponents***

Proof. Consider the collection of sets $V_c = \{g \in \mathbb{R}^n \,|\, \chi(g) \leq c\}$, then $V_{-\infty} = \{0\}$ and $V_\infty = \mathbb{R}^n$. In addition $V_a \subset V_b$ if $a \leq b$. Thus

$$\exists N \neq \emptyset \, \forall i \in N \, \exists r_i \in \mathbb{R} : V_{-\infty} \subsetneq V_{r1} \subsetneq \cdots \subsetneq V_\infty$$

Since V_{r_i} is a subspace, we have (where $k \geq 1$)

$$V_{r_i} = \operatorname{span}(v_1, \ldots, v_m) \subsetneq \operatorname{span}(v_1, \ldots, v_m, v_{m+1}, \ldots, v_{m+k}) = V_{r_{i+1}}$$

Clearly $\dim V_{-\infty} = 0$ and $\dim V_\infty = n$ and

$$0 = \dim V_{-\infty} < \dim V_{r_1} < \cdots < \dim V_\infty = n$$

But since dimension is an integer quantity, we have at most $n-1$ different r_i and therefore at most n finite Lyapunov exponents. □

Now we have established that there are at most n different Lyapunov exponents. In practice we are interested in the largest of them, since it is a suitable measure of the predictability (i.e. how fast the flows diverge in the worst case) of our system.

Definition 3.4. (Maximal Lyapunov exponent)

$$\chi_{\max}(X, x_0) = \max\{\chi(g) \,|\, g \in \mathbb{R}^n\}$$

However, even in simple examples the lim sup in equation 28 is difficult to compute, at best. Thus, we are interested under what conditions we can use a limit instead. Oseledets theorem answers this questions. In fact it also asserts that for sufficient regular differential equations the Lyapunov exponents are independent of the position for almost all x_0. Thus, our analysis of the Lyapunov exponents is effectly reduced to an n-number summary for a given system of differential equations.

Theorem 3.4. ***(Oseledets theorem)*** *If $\|X(x_0,t)\|$, $\left\|X(x_0,t)^{-1}\right\| \in L^1$ where $\|\cdot\|$ denotes the operator norm. Then, for almost all x and $u \neq 0 \in \mathbb{R}^n$ the limit*

$$\chi(X, x_0, u) = \lim_{t\to\infty} \frac{1}{t} \log \frac{\|X(x_0,t)u\|}{\|u\|}$$

exists and doesn't depend on x_0.

Proof. See [11]. □

The following two examples should illustrate some of the features of the Lyapunov exponent.

Example 3.2. Consider the simple example

$$\dot{x} = ax \qquad x : I \to \mathbb{R}$$

For any flow the fundamental solution is $X(t) = \mathrm{e}^{at}$. Then

$$\chi = \lim_{t\to\infty} \frac{1}{t} \log \frac{\|\mathrm{e}^{at} u\|}{\|u\|} = a$$

thus, we have only one Lyapunov exponent which is independent of x_0 and u. In this case $\chi_{\max} = a$. Note that if the Lyapunov exponent is positive the flows diverge, while if the Lyapunov exponent is negative they converge to some limit (in this case $\lim_{t\to\infty} \mathrm{e}^{ax} = 0$ since $a < 0$).

Example 3.3. Consider the non-linear differential equations

$$\dot{x} = x - x^3$$
$$\dot{y} = -y$$

Note that this system has 3 equilibria at $(0,0),(-1,0),(1,0)$. At this equilibria the Lyapunov exponents are trivial to compute. First, let us linearize our equations around the flow $\Phi(t,x_0)=(x(t),y(t))^T$

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} 1-3x^2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

Replacing x by the flow of our first equilibrium $x(t)=0$, i.e. the flow is constant and doesn't depend on t, we get

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

with the fundamental solution matrix

$$X = \begin{bmatrix} \mathrm{e}^t & 0 \\ 0 & \mathrm{e}^{-t} \end{bmatrix}$$

Using e_1, e_2 (the standard basis vectors), we get

$$\chi(0,0,e_1) = \lim_{n\to\infty} \frac{1}{t} \log \frac{\|Xe_1\|}{\|e_1\|} = 1$$
$$\chi(0,0,e_2) = \lim_{n\to\infty} \frac{1}{t} \log \frac{\|Xe_2\|}{\|e_2\|} = -1$$

Since there are at most $n=2$ Lyapunov exponents, we know all of them. This equilibrium, however, is atypical if we compare it with the result of the other two

$$\chi(-1,0,e_1) = -2$$
$$\chi(-1,0,e_2) = -1$$
$$\chi(1,0,e_1) = -2$$
$$\chi(1,0,e_2) = -1$$

However, by Oseledets theorem, we know that even for sufficient regular dynamical systems there might exist a set of measure zero on which the Lyapunov exponents don't agree with the Lyapunov exponents almost everywhere. It seems that $(0,0)$ belongs to that set.

Now consider an arbitrary flow through $(x,0)$. In this case we get by separation of variables (and if we assume $x\in(0,1)$)

$$x^2(t) = \frac{\mathrm{e}^{2t}}{\mathrm{e}^{2t}+1}$$

Inserting this into our linearized system we get

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} 1-\frac{3\mathrm{e}^{2t}}{\mathrm{e}^{2t}+1} & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

with the fundamental solution matrix

$$X = \begin{bmatrix} \frac{\mathrm{e}^{-2t}}{(1+\mathrm{e}^{-2t})^{3/2}} & 0 \\ 0 & \mathrm{e}^{-t} \end{bmatrix}$$

then

$$\chi(x,0,e_1) = -2$$
$$\chi(x,0,e_2) = -1$$

as expected.

It should be noted that (contrary to what the previous examples suggest) computing the Lyapunov exponent, in general, is a formidable task. Thus, one usually resorts to numerical algorithms (see e.g. [4]). We will continue this discussion in section 3.7.

3.3 From continuous to discrete - Poincaré maps

We consider differential equations of the form

$$\dot{x} = f(x), \qquad x : I \to \mathbb{R}^n \tag{29}$$

where we require $f \in \mathcal{C}^r$ for some $r \in \mathbb{N}$, $r \geq 1$.

Definition 3.5. (Poincaré map of a periodic orbit) Suppose we have a periodic flow (i.e. $\Phi(t + T, x_0) = \Phi(t, x_0)$). Let Σ be a hypersurface with normal $n(x)$ and $\langle f(x_0), n(x_0)\rangle \neq 0$ (i.e. Σ is transverse to f at x_0). Since $\Phi \in \mathcal{C}^r$, we can find an open set $V \subset \Sigma$, such that the flows starting in V returns to Σ in a time close to T, i.e.

$$P : V \to \Sigma,\ x \mapsto \Phi(\tau(x), x)$$

where $\tau(x)$ is the time of first return of the point x to Σ. We call P a **Poincaré map**.

We note that Poincaré maps can be defined for other types of orbits (e.g. for homoclinic orbits). There is no general method to choose Σ or to write down P explicitly (for a given vector field f). In three dimensions we can use numerical methods to construct a two-dimensional Poincaré map and analyze its behavior. However, for our purpose, the conceptual idea is important. A Poincaré map reduces an n dimensional continuous dynamical system to an $n-1$ dimensional discrete dynamical system, i.e. we can study certain properties of our original system by considering the associated discrete system.

3.4 The Smale horseshoe map

In the previous section we introduced a method (Poincaré maps) to study the behavior of an n dimensional continuous dynamical system by investigating an associated $n-1$ dimensional discrete dynamical system. Since three-dimensional continuous dynamical systems are of great practical importance (e.g. Lorenz equation, classical physics, ...), in this chapter, we investigate the behavior of a specific two-dimensional discrete system (the Smale horseshoe map). This system serves as a prototype for a number of properties we can find in continuous dynamical systems as well (this discussion is continued in section 3.7).
Consider an affine map $f : D \to \mathbb{R}^2$ where $D = [0, 1]^2$, which behaves as illustrated in figure 3. Then f

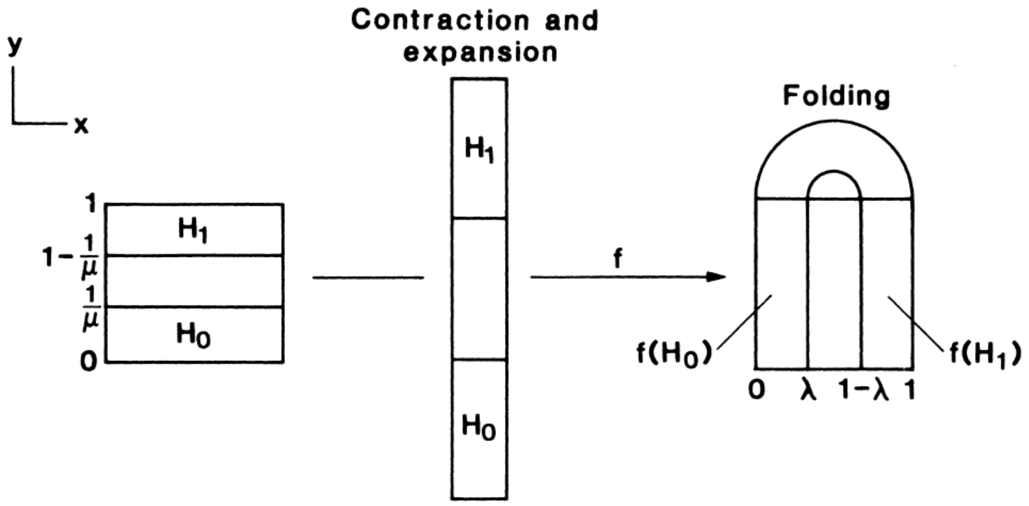

Figure 3: The Smale horseshoe map (this figure was taken from [14, p. 556])

on H_0 and H_1 is given by (see [14, p. 556])

$$f|_{H_0} : H_0 \to \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$f|_{H_1} : H_1 \to \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} -\lambda & 0 \\ 0 & -\mu \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ \mu \end{bmatrix}$$

The inverse map is depicted in figure 4

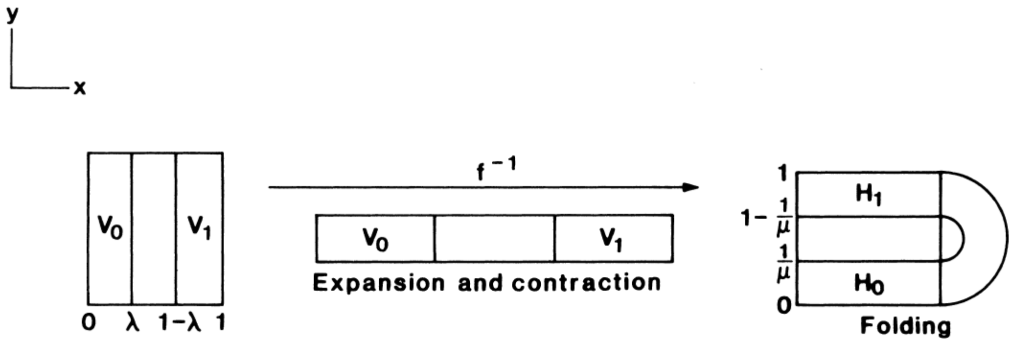

Figure 4: The inverse Smale horseshoe map (this figure was taken from [14, p. 557])

For our subsequent discussion we need the following definition.

Definition 3.6. A **vertical rectangle** is a rectangle such that the width parallel to the y-axis is 1. A **horizontal rectangle** is a rectangle such that the width parallel to the x-axis is 1.

The following lemma will also prove useful.

Lemma 3.2. ***(Mapping properties of*** f***)***

1. *Suppose* $V \in D$ *is a vertical rectangle, then* $f(V) \cap D$ *consists of precisely two vertical rectangles, one in* V_0 *and one in* V_1*. Their widths each are equal to* λ *times the width of* V*.*
2. *Suppose* $H \in D$ *is a horizontal rectangle, then* $f^{-1}(H) \cap D$ *consists of precisely two horizontal rectangles, one in* H_0 *and one in* H_1*. Their width each are equal to* $1/\lambda$ *times the width of* H*.*

Proof. Suppose we have a vertical rectangle V, i.e. we can write

$$V = (V \cap H_1) \cup (V \cap H_0) \cup V_{\text{rest}}$$

where $V_{\text{rest}} \cap (H_0 \cup H_1) = \emptyset$. Then clearly we get two rectangles in $f(H_0)$ and $f(H_1)$ respectively. The proof of *2.* uses the same principle. □

Now we are interested in the set of all points, denoted by Λ, which remain in D under all possible iterations of f; thus, we define

$$\Lambda = \bigcap_{n=-\infty}^{\infty} f^n(D)$$

We introduce the following notation: Let $S = \{0, 1\}$ be an index set, and let $s_i \in S$ for $i \in \mathbb{Z}$
First, $D \cap f(D)$ is given by (see the construction of f)

$$D \cap f(D) = \bigcup_{s_{-1} \in S} V_{s_{-1}} = \{p \in D \,|\, p \in V_{s_{-1}},\, s_{-1} \in S\}$$

where $V_{s_{-1}}$ is a vertical rectangle of width λ. Now we continue

$$\begin{aligned}
D \cap f(D) \cap f^2(D) &= D \cap f(D \cap f(D)) \\
&= D \cap f\left(\bigcup_{s_{-2}\in S} V_{s_{-2}}\right) \\
&= \bigcup_{s_{-2}\in S} D \cap f(V_{s_{-2}}) \\
&= \bigcup_{s_{-i}\in S,\, i=1,2} V_{s_{-1}} \cap f(V_{s_{-2}}) \\
&= \bigcup_{s_{-i}\in S,\, i=1,2} V_{s_{-1}s_{-2}}
\end{aligned}$$

Continuing this process k-times we get 2^k vertical rectangles, each of width λ^k, given by

$$\begin{aligned}
D \cap f(D) \cap \cdots \cap f^k(D) &= \bigcup_{s_{-i}\in S,\, i=1,\dots,k} V_{s_{-1}} \cap f(V_{s_{-2}\dots s_{-k}}) \\
&= \bigcup_{s_{-i}\in S,\, i=1,\dots,k} V_{s_{-1}\dots s_{-k}} \\
&= \{p \in D \mid f^{-i+1}(p) \in V_{s_{-i}},\ s_{-i} \in S, i = 1, \dots, k\}
\end{aligned}$$

Now we consider $k \to \infty$. Since a decreasing intersection of compact sets is non empty, we obtain an infinite number of vertical rectangles and the width of these rectangles is $\lim_{k\to\infty} \lambda^k = 0$.

Thus $\bigcap_{n=0}^{\infty} f^n(D)$ consists of an infinite number of vertical lines and we can label each such line uniquely by a binary infinite sequence.

We now repeat the whole process above with f^{-n} and horizontal rectangles. We then get an infinite number of horizontal lines which we label uniquely with a binary infinite sequence.

Now

$$\Lambda = \bigcap_{n=-\infty}^{\infty} f^n(D) = \left[\bigcup_{n=-\infty}^{0} f^n(D)\right] \cap \left[\bigcup_{n=0}^{\infty} f^n(D)\right]$$

which consists of an infinite set of points (each vertical line intersects each horizontal line in a unique point). Furthermore, each $p \in \Lambda$ can be labeled uniquely by a binary bi-infinite sequence which we form by concatenating the sequences associated with the respective lines which intersect to give p.

Thus, we have a well-defined map (where Σ is the set of all binary bi-infinite sequences)

$$\phi : \Lambda \to \Sigma,\ p \mapsto \{\dots s_{-k} \dots s_{-1}.s_0 \dots s_k \dots\}$$

The dot is used to separate the two infinite sequences. Now we have established some sort of connection between the invariant set of the Smale Horseshoe map and all binary bi-infinite sequences. In the next chapter we will investigate the exact nature of this relationship in more detail.

3.5 Symbolic dynamics

First, we define a metric on Σ

$$d(s, \overline{s}) = \sum_{i=-\infty}^{\infty} \frac{\delta_i}{2^{|i|}}, \qquad \delta_i = \begin{cases} 0 & s_i = \overline{s}_i \\ 1 & s_i \neq \overline{s}_i \end{cases}$$

Then we define the shift map as

Definition 3.7. (Shift map)

$$\sigma : \Sigma \to \Sigma,\ \{\dots s_{-k} \dots s_{-1}.s_0 s_1 \dots s_k \dots\} \mapsto \{\dots s_{-k} \dots s_{-1} s_0.s_1 \dots s_k \dots\}$$

Now let us investigate the dynamics of the shift map.

Theorem 3.5. *The shift map σ has*

1. *a countable infinity of periodic orbits.*
2. *an uncountable infinity of nonperiodic orbits.*

Proof.

1. Consider periodic orbits in Σ defined as
$$\overline{\{s_0 s_1 \dots s_k\}} = \{\dots s_0 \dots s_k s_0 \dots s_k . s_0 \dots s_k \dots\}$$
For the set of all such periodic orbits, denoted by Q, it holds that $Q \subset \mathbb{Q} \times \mathbb{Q}$, but since Q is clearly infinite we get our result as desired.
2. Given a non-repeating sequence in Σ we get
$$\{\dots s_{-n} \dots s_{-1} . s_0 \dots s_n \dots\} \cong \{.s_0 s_1 s_{-1} s_2 s_{-2} \dots\} \cong \mathbb{I} \cap [0,1]$$
since the set of irrational numbers in $[0,1]$ is uncountable our result follows.

□

We will now apply our results from symbolic dynamics to the Smale horseshoe. To do this we first proof that there is a homeomorphism between the two systems.

Theorem 3.6. $\phi : \Lambda \to \Sigma$ *is a homeomorphism.*

Proof. Bijectivity follows from our construction of the binary bi-infinite sequences. To proof continuity we need to show
$$\forall p \in \Lambda \exists \epsilon > 0 \exists \delta > 0 : |p - \overline{p}| < \delta \Longrightarrow d(\phi(p), \phi(\overline{p})) < \epsilon$$
Let $\epsilon > 0$ be given. If $d(\phi(p), \phi(\overline{p})) < \epsilon$, there must be some $N(\epsilon) \in \mathbb{N}$ such that if
$$\phi(p) = \{\dots s_{-n} \dots s_{-1} . s_0 \dots s_n \dots\}$$
$$\phi(\overline{p}) = \{\dots \overline{s}_{-n} \dots \overline{s}_{-1} . \overline{s}_0 \dots \overline{s}_n \dots\}$$
then $s_i = \overline{s}_i$, $i = 0, \pm 1, \dots, \pm N$. Thus by construction we have $p, \overline{p} \in H_{s_0 \dots s_N} \cap V_{s_{-1} \dots s_{-N}}$. This rectangle has width λ^N and height $1/\mu^{N+1}$. Then
$$|p - \overline{p}| \leq \left(\lambda^N + \frac{1}{\mu^{N+1}} \right)$$
We choose
$$\delta = \lambda^N + \frac{1}{\mu^{N+1}}$$
which suffices to proof continuity. Since continuous bijections from compact sets into Hausdorff spaces are homeomorphism (see [14, p. 571]) this concludes our proof. □

Since we have a homeomorphism, we now can use the fact that
$$\left(\phi^{-1} \circ \sigma \circ \phi \right)(p) = f(p), \qquad \forall p \in \Lambda$$
i.e., the dynamics of f is given by the dynamics of σ. E.g. an orbit of σ corresponds to an orbit of f and vice-versa.

Theorem 3.7. *The Smale horseshoe has*

1. *a countable infinity of periodic orbits.*

2. an uncountable infinity of non-periodic orbits.

Proof.

1. Suppose we have a periodic orbit in Σ, i.e. $\exists k \in \mathbb{N}$

$$w := \overline{\{s_0 s_1 \dots s_k\}} = \sigma^k \left(\overline{\{s_0 s_1 \dots s_k\}} \right)$$

then we use the homeomorphism to get

$$f^k(p) = \phi^{-1} \circ \sigma^k(w) = \phi^{-1}(w) = p$$

for $p \in \Lambda$. Reversing the argument, we get a bijection between the periodic orbits in Σ and the periodic orbits in Λ, as desired.

2. Suppose we have a non-periodic orbit $w \in \Sigma$, i.e.

$$\forall k : \sigma^k(w) \neq w$$

Since ϕ is a bijection we get

$$\forall k : \phi^{-1} \circ \sigma^k(w) \neq \phi^{-1}(w)$$

from which

$$\forall k : f^k(p) \neq p$$

for $p \in \Lambda$. Reversing the argument we get a bijection between the non-periodic orbits in Σ and Λ, as desired.

□

3.6 Chaos in discrete systems

Now we are finally able to define and proof existence to a major ingredient of deterministic chaos

Definition 3.8. (Sensitive dependence on initial conditions). A dynamical system exhibits sensitive dependence on initial conditions if

$$\exists c \in \mathbb{R}_{>0} \, \forall \epsilon > 0 \, \exists U(p) \, \exists q \in U(p) \, \exists N \in \mathbb{N} \, \forall n > N : \, d(f^n(p), f^n(q)) > c$$

Now we proof that the Smale horseshoe exhibits this behavior.

Theorem 3.8. *The Smale horseshoe exhibits sensitive dependence on initial conditions*

Proof. First, for any $p \in \Lambda$ we have

$$\phi(p) = \{\dots s_{-n} \dots s_{-1}.s_0 \dots s_n \dots\}$$

We consider an ϵ-neighborhood of p, then the corresponding neighborhood of $\phi(p)$ includes a set of sequences

$$\overline{s} = \{\dots \overline{s}_{-n} \dots \overline{s}_{-1}.\overline{s}_0 \dots \overline{s}_n \dots\}$$

such that $s_i = \overline{s}_i$ for all $|i| \geq N$ for some $N \in \mathbb{N}$. Then we have some $\overline{s}$ such that $s_{N+1} \neq \overline{s}_{N+1}$. Now we apply σ^N, and clearly

$$d(\sigma^N(s), \sigma^N(\overline{s})) \geq 1 > \epsilon$$

using ϕ^{-1} our result follows. □

3.7 Chaos in continuous systems

In the previous sections we went to great length to develop the Smale horseshoe as well as to show that it has certain properties (most notable sensitivity on initial conditions) which are associated with chaotic behavior. However, it isn't clear how we can apply this knowledge to our main objective of study, i.e. how to analyze *continuous* dynamical systems.

In this section we will develop the definition of chaos in continuous dynamical systems as well as give a short overview of the proof techniques we can employ to associate the behavior of such a system with a chaotic invariant set (i.e. a Smale horseshoe). Unfortunately, to proof chaotic behavior even in simple systems (i.e. f is a polynomial of order 2) is a highly formidable task. That the Lorenz equation (with $\sigma = 10$, $\beta = \frac{8}{3}$, $\rho = 28$, see [2, p. 47-48])

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= x(\rho - z) - y\\ \dot{z} &= xy - \beta z\end{aligned}$$

is indeed chaotic is still not proven [6, p. 584-585] (however, there exists proof for a number of similar systems), even though the equation was known and analyzed numerically in a paper by Lorenz (see [10]) in 1963. Thus, a complete treatment of chaos in continuous dynamical systems is beyond the scope of this bachelor thesis.

Now let us start with some necessary definitions:

Definition 3.9. (Compact invariant set) We say a compact set $\Lambda \subset \mathbb{R}^n$ is invariant under $\Phi(t, x)$ if

$$\Phi(t, \Lambda) \subset \Lambda, \qquad \forall t \in I$$

Definition 3.10. (Topological Transitivity) A compact invariant set Λ is topologically transitive if

$$\forall U, V \text{ open} \subset \Lambda\, \exists t \in I : \Phi(t, U) \cap V \neq 0$$

Definition 3.11. (Sensitive dependence on initial conditions) A flow Φ is said to have sensitive depence on initial conditions on an invariant set Λ if

$$\exists \epsilon > 0\, \forall x \in \Lambda\, \forall U(x)\, \exists y \in U, t \in I : |\Phi(t, x) - \Phi(t, y)| > \epsilon$$

This means that for every point $x \in \Lambda$ we have at least one point that is arbitrarily close to x but after some time it diverges such that the distance is at least ϵ.

Definition 3.12. (Chaotic set) A compact set Λ is said to be chaotic if

1. $\Phi(t, x)$ has sensitive depence on initial conditions in Λ
2. $\Phi(t, x)$ is topologically transitive on Λ

That the set Λ needs indeed to be compact (for a sensible definition of chaos) can be seen from the simple linear system $\dot{x} = ax$ we considered in 3.2. Even though the flows diverge exponentially we wouldn't consider the system chaotic. In the Lorenz equation, however, the flows diverge exponentially, even though it can be shown that the flow approaches and stays inside a compact ellipse. Next let us introduce the notion of a *strange attractor*.

Definition 3.13. (Attractor) A compact invariant set Λ is called an attractor if there is some neighborhood $U \supset \Lambda$ such that

$$\Phi(t, U) \subset U, \; \bigcap_{t \in I} \Phi(t, U) = \Lambda$$

We call U a **trapping region**.

Definition 3.14. (Strange attractor) Suppose Λ is an attractor, we call it strange if Λ is chaotic.

The Lorenz equation is a prototypical example of a system which is suspected to posses a strange attractor. The difficulty is inherent in actually proving this. A method (in fact the most common method) is to choose an appropriate Poincaré map near an orbit of our continuous dynamical system and proof, that this map exhibits the same homeomorphism to symbolic dynamics as did the Smale horseshoe.

This method has been formalized and is known as the *Conley-Moser conditions*. The interested reader is advised to read e.g. [14, Chap. 26,27,30] as an introduction.

The question remains how one determines, short of a mathematical proof, if a system is chaotic (This is of vital importance since deriving predictions from chaotic systems is more difficult and might require a different set of tools). We introduced the Lyapunov exponent in section 3.2. A positive $\chi_{\max}$ (the maximal Lypaunov exponent) is a strong indication of chaotic behavior given we can show that the solution stays inside/approaches a compact set.

The Lyapunov exponents of the Lorenz equation (calculated numerically in [4]) are

$$\chi \approx 0.9,\ 0,\ -14.6$$

They strongly suggest the existence of chaotic behavior in this system (since $\chi_{max} = 0.9 > 0$). Let us conclude our discussion with the following example.

Example 3.4. In [15] chaotic behavior is analyzed for the following third order equation (the linear Linz-Sprott equation).

$$\dddot{x} + a\ddot{x} + b\dot{x} - |x| + 1 = 0$$

with $a = 0.6$, $b = 1$. The proof proceeds in two steps

1. Find a suitable Poincaré map (in this case near a quasi-homoclinic orbit) for the system by analyzing it numerically/geometrically.
2. Prove that this Poincaré map has dynamics similar to those of the Smale horseshoe.

Where [15] mainly differs from our discussion in that numerical methods with error estimation are used instead of deriving an analytic proof (This is also known as a *computer assisted proof*).

4 Conclusion

This bachelor thesis gives an overview of some methods used to analyze non-linear differential equations as well as exemplifies a very important type of behavior, chaos, that occurs in such systems.

For the reader who is interested in perturbative methods, KAM theory is the next logical step (see e.g. [9] or [14, Chap. 14]). For the reader more interested in qualitative and geometric analysis, [5] is recommended.

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