(6.1) Let K be a triangle with vertices P, Q, R, let ψ be a polynomial of degree at most one, and denote by $|K|$ the area of K. Show that

(a) $$\int_K \psi \, \mathrm{d}(x,y) = \frac{|K|}{3}\big(\psi(P) + \psi(Q) + \psi(R)\big)$$

(b) $$\int_K \psi \, \mathrm{d}(x,y) = |K|\psi(P_K), \quad P_K = \frac{1}{3}(P + Q + R)$$

(6.2) Solve the elliptic problem

$$-\Delta u = 1$$

on $\Omega = (0,1)^2$ with homogeneous Dirichlet boundary conditions. Use a regular triangulation with right-angled triangles. Compare the stiffness matrix with the corresponding matrix in a finite difference approximation.

(6.3) Repeat the above exercise for

$$\partial_x\big(a(x,y)\partial_x u\big) + \partial_y\big(b(x,y)\partial_y u\big) = -1$$

with $a(x,y) = 1 + x$ and $b(x,y) = 2 - xy$.

(6.4) Consider the elliptic problem

$$-u''(x) = f(x), \qquad 0 < x < 1$$

with boundary conditions $u(0) = 2$ and $u'(1) = 1$. Give a weak formulation of this problem and explain its numerical solution with linear finite elements. In particular, specify V_h and a basis of V_h.

References:

S. Larsson, V. Thomee, Partial differential equations with numerical methods