(5.1) We discuss once more Exercise 4.3.

(5.2) Let $T = \{(x,y)\ ;\ 0 \le x \le h,\ 0 \le y \le h - x\}$. For a continuous function w, we denote $\|w\|_\infty = \sup_{(x,y)\in T} |v(x,y)|$. Further let $I_h w$ be the (unique) polynomial of degree one that interpolates w at the vertices of T.

Show that there exists a constant $C > 0$ such that

$$\left\|I_h v - v\right\|_\infty \le Ch^2 \left(\|v_{xx}\|_\infty + \|v_{xy}\|_\infty + \|v_{yy}\|_\infty\right)$$

for all $v \in C^2(T)$.

Hint. Express $I_h v$ in terms of v and use Taylor's formula.

(5.3) Recall Green's formulas (from Analysis 2, e.g.) and verify that

$$-\int_\Omega v\,\Delta u\ \mathrm{d}(x,y) = \int_\Omega \nabla v \cdot \nabla u\ \mathrm{d}(x,y)$$

for $u \in C^2(\overline{\Omega})$ and $v \in C_0^1(\overline{\Omega})$.

(5.4) Show that the solution of the elliptic problem

$$-\Delta u = f, \quad u|_{\partial\Omega} = 0$$

does not automatically inherit the smoothness of f. For example, even for $f \in C^\infty$ the solution u does not need to be twice continuously differentiable.

Hint. Consider $\Omega = (0,1)^2$ and $f = 1$. Compute $u_{xx}(0,0)$.

(5.5) For a symmetric $n \times n$ matrix A and a vector $b \in \mathbb{R}^n$, consider the bilinear form

$$a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} : (u,v) \mapsto u^T A v$$

and the linear form

$$\ell : \mathbb{R}^n \to \mathbb{R} : v \mapsto b^T v.$$

Formulate and prove the Lax–Milgram lemma for this situation. Which additional property must be required for a? Show that $a(u,u) = \|u\|_a^2$ defines a norm, the so-called energy norm.

(5.6) Let V be a Hilbert space and $a : V \times V \to \mathbb{R}$ a symmetric and bounded bilinear form. If a is coercive then the energy norm $\|\cdot\|_a$ is equivalent to the norm of V. Verify this statement.

References:

S. Larsson, V. Thomee, Partial differential equations with numerical methods