(3.1) Solve the linear advection problem

$$u_t + u_x = 0, \qquad 0 \le x \le 1, \quad 0 \le t \le 0.5$$

on a computer with the upwind scheme

$$u_j^{n+1} = u_j^n - \tfrac{\tau}{h}\big(u_j^n - u_{j-1}^n\big)$$

and the Lax–Wendroff scheme

$$u_j^{n+1} = u_j^n - \tfrac{\tau}{2h}\big(u_{j+1}^n - u_{j-1}^n\big) + \tfrac{\tau^2}{2h^2}\big(u_{j+1}^n - 2u_j^n + u_{j-1}^n\big)$$

respectively. You may choose $\tau/h = 0.5$. Try different values of h, e.g. $h = 0.01$.

(a) Choose periodic boundary conditions (i.e. $u(0,t) = u(1,t)$) and a smooth initial value, e.g. $u_0(x) = \sin(2\pi x)$.

(b) Choose an inflow condition at $x = 0$ (i.e. $u(0,t)$ given) and the non-smooth initial value $u_0(x) = 1$ for $x = 0$ and $u_0(x) = 0$ for $x > 0$. The choice $u(0,t) = 1$ models a Heaviside initial value.

(c) Plot the solutions and the errors.

(d) Study numerically the Gibbs phenomenon of the Lax–Wendroff scheme. Determine in particular the maximum of the numerical solution and its position as a function of h for $t = 0.5$.

(3.2) Solve Burger's equation

$$u_t + uu_x = 0, \qquad 0 \le x \le 1, \quad 0 \le t \le 0.5$$

on a computer with Riemann initial data

$$u_0(x) = \begin{cases} 1.2 & x < 0 \\ 0.4 & x \ge 0 \end{cases}$$

for $\tau/h = 0.5$ and various values of h using

(a) the upwind scheme;

(b) the conservative upwind scheme.

Determine numerically the speed of the shock front (as a function of h) and compare it with the exact result.

(3.3) Verify that the Burgers equation with initial data

$$u_0(x) = \begin{cases} 1 & x < 0 \\ 0 & x \geq 0 \end{cases}$$

has the weak solution

$$u(t,x) = \begin{cases} 1 & x \leq -t/2 \\ -2 & -t/2 \leq x \leq 0 \\ 2 & 0 \leq x \leq t \\ 0 & t \leq x \end{cases}$$

that consists of three shocks. Does this solution satisfy the Lax entropy condition?

(3.4) Find the unique weak solution of Burgers' equation with initial data

$$u_0(x) = \begin{cases} 2 & x < 0 \\ 1 & 0 \leq x < 2 \\ 0 & x \geq 2 \end{cases}$$

that satisfies the entropy condition.

(3.5) Compute the solution of Exercise 3.4 with the conservative upwind scheme.

(3.6) Verify that the upwind flux

$$F(v,w) = f(v) \qquad \text{for } f' > 0$$

is consistent, i.e., it satisfies the two conditions given in the lecture.

References:

R.J. LeVeque, Numerical methods for conservation laws, 1999