Exercises 20.10.2014

2.1) Show using von Neumann stability analysis that the numerical method given by (for $v > 0$)

$$u_i^{n+1} = u_i^n - \frac{v\tau}{h}\left(u_i^n - u_{i-1}^n\right)$$

is stable under the assumption that

$$\frac{v\tau}{h} \leq 1.$$

2.2) Verify by implicit differentiation that

$$u(t,x) = u_0(x - tu(t,x))$$

gives a solution to Burgers' equation

$$\partial_t u(t,x) + u(t,x)\partial_x u(t,x) = 0.$$

2.3) Apply the method of characteristics to obtain a solution of

$$\partial_t u(t,x) + x\partial_x u(t,x) = \sin x.$$

2.4) Apply the method of characteristics to obtain a solution of Burgers' equation with the initial value

$$u_0(x) = x.$$

2.5) Implement the first and second order upwind schemes for the advection and Burgers' equation in a programming language of your choice and apply them to the following initial value

$$u_0(x) = \text{erf}(\lambda(x - 0.4)) - \text{erf}(\lambda(x - 0.6)),$$

where erf is the error function. Conduct simulations for a number of different τ, h, and $\lambda \geq 20$ on the domain $[0,1]$. What are your conclusions?

2.6) Approximate the solution of

$$u(t,x) = u_0(x - tu(t,x))$$

by programming the corresponding fixed-point iteration for some initial values of your choice. Plot the error as a function of the number of iterations conducted.

2.7) Argue that the map of the previous exercise is a contraction (for t sufficiently small).

References

E. Zauderer, Partial differential equations of applied mathematics, 1988

R.J. LeVeque, Numerical methods for conservation laws, 1999