Abstracts

A comparison of triple jump and Suzuki fractals for obtaining high order from an almost symmetric Strang splitting scheme

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We consider the time discretization of ordinary and partial differential equations. More specifically, we assume that the considered problem can be written as the following abstract Cauchy problem

$$u' = Au + B(u), \qquad u(0) = u_0. \tag{1}$$

In this context splitting methods can be applied if the partial flows generated by A and B have an analytical representation, or if an efficient algorithm for finding their exact solution is known. However, the assumption that the partial flow generated by the nonlinear operator B can be computed exactly is usually a very strong requirement.

To remedy this deficiency of classic splitting methods, we propose and analyze splitting schemes which approximate the partial flow generated by B by that of an inhomogeneous linear differential equation. That is, we consider the linearized problem given by

$$v' = b(u_\star)v + d, \tag{2}$$

where we assume that, once a value $u_\star$ is substituted, the flow corresponding to (2) can be computed efficiently.

In this context, we can formulate the (classic) Strang splitting scheme as follows

$$M_\tau(u_0) = \mathrm{e}^{\frac{\tau}{2}A}\varphi_\tau^{b(u_{1/2})}(\mathrm{e}^{\frac{\tau}{2}A}u_0), \qquad u_{1/2} = \varphi_{\frac{\tau}{2}}^{b(u_0)}(\mathrm{e}^{\frac{\tau}{2}A}u_0), \tag{3}$$

where the flow corresponding to equation (2) is denoted by $\varphi_t^{b(u_\star)}$; τ is the step size. The Strang splitting above is no longer symmetric. The lacking symmetry of the method does not severely affect performance; however, if composition is used as a means to construct higher order methods, symmetry is a desirable property.

To remedy the lack of symmetry we consider the symmetric scheme $u_1 = S_\tau(u_0)$ given by the solution of the following implicit equation

$$u_1 = \mathrm{e}^{\frac{\tau}{2}A} \circ \varphi_{\frac{\tau}{2}}^{b(u_1)}(u_{1/2}).$$

Note that solving this equation is computationally not attractive in practice. Therefore, we proposed to employ fixed-point iterations to compute an approximation to u_1 (see [1] and [2]). The resulting one-step methods are not symmetric but they are symmetric up to a given order (a precise definition is given in [1]). Thus, it is possible to construct composition methods of arbitrary (even) order.

In [1] and [2] we have performed numerical simulations for a variety of ordinary and partial differential equations. To obtain methods of order four we have used the well-known triple jump composition. However, at least for some ODEs, the so called Suzuki fractals give better performance (see, e.g., [3]). In this case, five

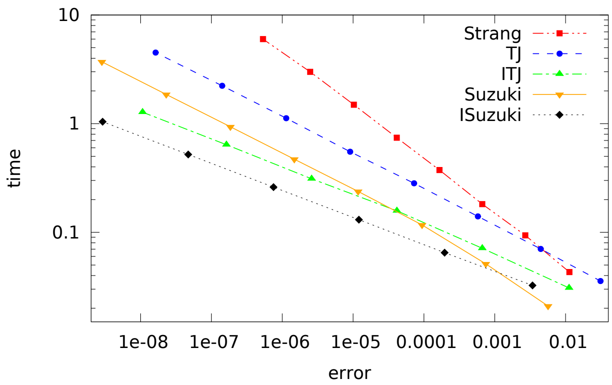

Figure 1. Work-precision plot (using the discrete infinity norm) for a charged particle in an inhomogeneous magnetic field integrated up to $T = 100$.

Strang splitting schemes have to be combined to obtain a scheme of order four (as compared to the three used in the triple jump approach). However, the error constant is expected to be smaller in the case of Suzuki fractals. Furthermore, the size of the negative time steps is significantly reduced compared to the triple jump scheme.

First, let us consider the ODEs describing a charged particle in an inhomogeneous magnetic field (more detailed information is provided in [1]). From the results shown in Figure 1, we observe that the iterated method (as described above) using the Suzuki fractals is clearly the method of choice for that particular problem. In [1] we have already shown that the splitting approach using the iterated triple jump scheme yields superior performance as compared to a fourth order Runge–Kutta method.

Second, we consider the parabolic Brusselator system (see [2]). In this case it is not possible to take negative time-steps. However, similar to the triple jump scheme, complex coefficients (with positive real part) can be determined that, using five compositions, yield a symmetric method of fourth order. While we observe from Figure 2 that the iterated triple jump schemes gives superior performance for medium to high precision (or equivalently long integration times), no additional benefit is observed by using the Suzuki fractals (in fact the run time for a given error is almost identical for the triple jump and the Suzuki fractal based methods).

Our last example is the hyperbolic KdV equation (see [2]). In this case, the Suzuki fractal method (that uses the classic Strang splitting and not its iterated version) already shows order four over some range in step size and thus, due to the low computational cost, proves to be the most efficient method (see Figure 3). We further note that the iterated schemes (both based on the Suzuki fractal as well as on the triple jump scheme) show similar performance.

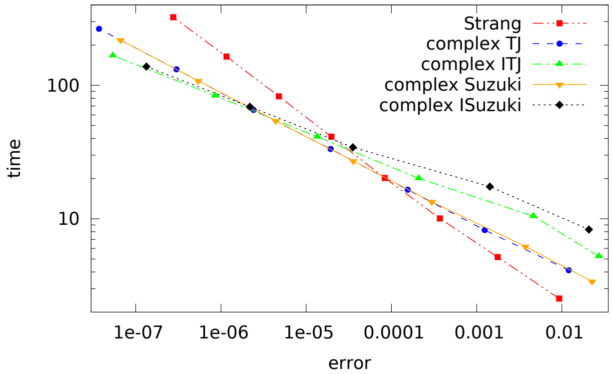

FIGURE 2. Work-precision plot (using the discrete infinity norm) for the Brusselator integrated up to $T = 0.25$ with 2^{10} grid points per dimension (in total we have 2^{21} grid points).

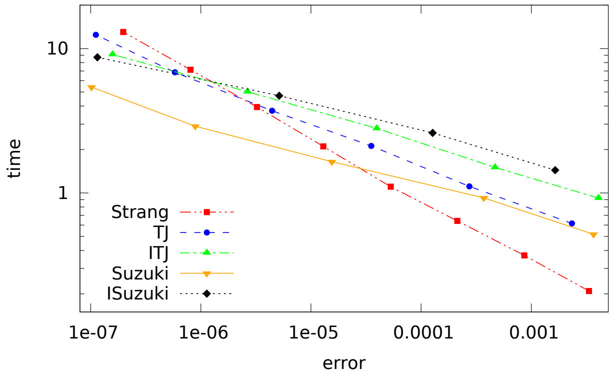

FIGURE 3. Work-precision plot (using the discrete infinity norm) for the KdV equation with the sech soliton initial value integrated up to $T = 0.05$ with 2^{10} grid points.

In addition, work-precision plots for a number of problems (including the Van der Pol oscillator) are given in [4].

REFERENCES

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