Sets containing a point of differentiability of every Lipschitz function.

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Sets containing a point of differentiability of every Lipschitz function.

A set $S \subseteq \mathbb{R}^d$ is called a *universal differentiability set* if S contains a point of differentiability of every Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$.

A set $E \subseteq \mathbb{R}^d$ is called a *non-universal differentiability set* if there exists a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ such that f is nowhere differentiable in E.

Examples.

- By Rademacher's theorem, any subset of $\mathbb{R}^d$ of positive Lebesgue measure is a universal differentiability set.
- Universal differentiabilty sets in $\mathbb{R}$ are precisely the sets of positive Lebesgue measure.

Existence of exceptional universal differentiability sets.

- (Preiss 1990) For $d \geq 2$, there exist universal differentiability sets in $\mathbb{R}^d$ with Lebesgue measure zero.
- (Doré, Maleva 2011) For $d \geq 2$, $\mathbb{R}^d$ contains compact universal differentiability sets with Hausdorff dimension one.
  Moreover, every universal differentiability set in $\mathbb{R}^d$ has Hausdorff dimension at least one.
- However each of the universal differentiability sets given above have the maximal possible Minkowski dimension d.

Differentiability inside sets of Minkowski dimension one.

Definition

Given a bounded subset A of $\mathbb{R}^d$ and $\epsilon > 0$, we define $N_\epsilon(A)$ as the minimal number of balls of radius ϵ required to cover the set A. The (upper) Minkowski dimension of A is then defined by

$$\dim_M(A) = \inf\left\{ s > 0 \quad : \quad \limsup_{\epsilon \to 0} N_\epsilon(A)\epsilon^s = 0 \right\}.$$

Theorem (D., Maleva 2013)

For $d \geq 1$, $\mathbb{R}^d$ contains compact universal differentiability sets with Minkowski dimension one.

Idea of the proof

Theorem (D. 2013, Doré, Maleva 2009)

Let $(U_\lambda)_{\lambda\in(0,1]}$ be a family of closed subsets of $\mathbb{R}^d$ satisfying the following conditions:

1. $U_{\lambda_1} \subseteq U_{\lambda_2}$ *whenever* $\lambda_1 \leq \lambda_2$,
2. *For all* $\lambda \in (0,1)$, $\psi \in (0, 1-\lambda)$ *and* $\eta > 0$ *there exists a number* $\delta_0 = \delta_0(\lambda, \psi, \eta) > 0$ *such that for all* $x \in U_\lambda$, $e \in S^{d-1}$ *and* $\delta \in (0, \delta_0)$ *there exists* $x' \in \mathbb{R}^d$ *and* $e' \in S^{d-1}$ *such that* $\|x' - x\| \leq \eta\delta$, $\|e' - e\| \leq \eta$ *and* $[x', x' + \delta e'] \subseteq U_{\lambda+\psi}$.

Then each set U_λ is a universal differentiability set.

Differentiability inside sets of Minkowski dimension one.

Theorem (D., Maleva 2013)

Let $d \geq 2$ and $S \subseteq \mathbb{R}^d$ be a set of finite 1-dimensional Hausdorff measure

$$\mathcal{H}^1 = \liminf_{\epsilon \to 0+} \left\{ \sum diam(S_i) \quad : S \subseteq \bigcup_{i=1}^{\infty} S_i \quad and \quad diam(S_i) \leq \epsilon \right\}.$$

Then S is a non-universal differentiability set.

- In particular any universal differentiability set U satisfies $\liminf_{\epsilon \to 0} N_\epsilon(U)\epsilon^1 = \infty$.

Connection to porosity.

Definition

A subset P of $\mathbb{R}^d$ is called porous if there exists $c \in (0,1)$ such that for every $x \in P$ and every $\epsilon > 0$ there exists $h \in \mathbb{R}^d$ with $\|h - x\| \leq \epsilon$ and $B(h, c\,\|h - x\|) \cap P = \emptyset$. P is called σ-porous if P can be expressed as a countable union of porous sets.

- Every porous subset of $\mathbb{R}^d$ is a *non-universal differentiability set.*
- (Kirchheim, Preiss, Zajíček, 2001) Every σ-porous subset of $\mathbb{R}^d$ is a *non-universal differentiability set.*

Can a universal differentiability set be decomposed?

Given a universal differentiability set $S \subseteq \mathbb{R}^d$, is it possible to write $S = A \cup B$ where A and B are non-universal differentiability sets?

Equivalently: Given a universal differentiabilty set $S \subseteq \mathbb{R}^d$ is it possible to find a pair of Lipschitz functions $f, g : \mathbb{R}^d \to \mathbb{R}$ such that f and g have no common points of differentiability in S.

Simultaneous differentiability of Lipschitz functions.

1. (Lindenstrauss, Tišer, Preiss, 2012) Every pair (f, g) of Lipschitz functions on a Hilbert space have a common point of differentiability.
2. **Open Question**: Does every triple (f, g, h) of Lipschitz functions on a Hilbert space have a common point of differentiability?

Can a universal differentiability set be decomposed?

Does there exist a unversal differentiability set $S \subseteq \mathbb{R}^d$ such that $S = A \cup B$ where A and B are non-universal differentiability sets?

Equivalently: Does there exist a universal differentiabilty set $S \subseteq \mathbb{R}^d$ for which it is possible to find a pair of Lipschitz functions $f, g : \mathbb{R}^d \to \mathbb{R}$ such that f and g have no common points of differentiability in S.

The answer is yes.

1. (Csörnyei, Preiss, Tišer, 2004), (Alberti, Csörnyei, Preiss, 2010) Case $d = 2$.
2. (Preiss, Speight 2014)(Csörnyei, Jones) Case $d > 2$.

Structural results for universal differentiability sets.

Theorem (D., 2013)

1. *Let $S = A \cup B \subseteq \mathbb{R}^d$ be a universal differentiability set where A is a closed subset of S. Then either A or B is a universal differentiability set.*
2. *Let $S = \bigcup_{i=1}^{\infty} A_i \subseteq \mathbb{R}^d$ be a universal differentiability set, where each A_i is a closed subset of S. Then at least one A_i is a universal differentiability set.*

- Recall: (Csörnyei, Preiss, Tišer, 2004) There exists a universal differentiability set $S = A \cup B \subseteq \mathbb{R}^d$ such that both A and B are non-universal differentiability sets.

Structural results for universal differentiability sets.

Theorem (D., 2014)

1. *Let $S = A \cup B \subseteq \mathbb{R}^d$ be a universal differentiability set where A is a closed subset of S. Then either A or B is a universal differentiability set.*
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- (Csörnyei, Preiss, Tišer, 2004) There exists a universal differentiability set $S = A \cup B \subseteq \mathbb{R}^d$ such that A is a G_δ set and both A and B are non-universal differentiability sets.

The kernel of a universal differentiability set.

Theorem (D., 2014)

Let S be a universal differentiability set and define

$$\ker(S) = S \setminus \{x \in S : \exists r > 0 \ s.t.\ B(x,r) \cap S \ is\ a\ non\text{-}UDS\}.$$

Then,

1. $\ker(S)$ *is a universal differentiability set.*
2. $\ker(\ker(S)) = \ker(S)$.

Open questions.

1. Does every universal differentiability set contain a closed universal differentiability set?
2. Does every subset of $\mathbb{R}^d$ with positive Lebesgue measure contain a universal differentiability set of Lebesgue measure zero?

Thank you for listening.