The Kakeya Problem

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The Kakeya problem is a famous open problem and the motivation for a diverse range of modern research in mathematics. A subset of $\mathbb{R}^n$ is called a Besicovitch set if it contains a line segment of length one in every direction. Clearly the unit ball in $\mathbb{R}^n$ is an example of a Besicovitch set. However, it is known that Besicovitch sets exist with Lebesgue measure zero, i.e. much smaller than the unit ball. The Kakeya problem focuses on the question of how large a Besicovitch set has to be.

There are finer tools than the Lebesgue measure for distinguishing between the sizes of sets, most notably the Hausdorff and Minkowski dimensions. It is therefore natural to ask whether Besicovitch sets may be smaller than Lebesgue measure zero. The Kakeya conjecture states that this is not possible and that Besicovitch sets have to be quite large. More precisely, the conjecture states that a Besicovitch set in $\mathbb{R}^n$ must have Hausdorff and Minkowski dimension n. In other words, according to the Hausdorff and Minkowski dimensions, Besicovitch sets should have the same size as the unit cube $[0,1]^n$ in $\mathbb{R}^n$. The validity of the Kakeya conjecture is known in dimensions $n = 1$ and 2, but unknown for all higher dimensions.

The aim of this project is to provide a detailed introduction to the Kakeya problem, presenting the necessary theory of the Hausdorff and Minkowski dimensions. The project should include proofs of the existence of Besicovitch sets of Lebesgue measure zero and the Kakeya conjecture in dimensions 1 and 2, before a discussion of the situation in higher dimensions.