Embedding Finite Metric Spaces into Normed Spaces.

Themenvorschlag zur Bachelorarbeit, Sommersemester 2016.

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Given a finite metric space \((M, d)\), it is natural to ask whether we can imagine \(M\) as a collection of points in some Euclidean space. In other words, does there exist an embedding \(i: M \rightarrow \mathbb{R}^n\) such that \(\|i(x) - i(y)\| = d(x, y)\) for all \(x, y \in M\). In general, this is too much to ask for. For example, the following two metric spaces cannot be embedded into any Euclidean space:

Both of the metric spaces above contain 4 points depicted as vertices of a graph. The distance between any two points \(x\) and \(y\) is defined as the minimal length of a path connecting \(x\) and \(y\).

If it is not possible to embed \(M\) into a Euclidean space \(\mathbb{R}^n\), we may allow ourselves the additional freedom to stretch and compress the distances in \(M\) within a certain limit. Put differently, we ask whether there is some constant \(K > 1\) and a mapping \(f: M \rightarrow \mathbb{R}^n\) such that

\[
\frac{1}{K} d(x, y) \leq \|f(x) - f(y)\| \leq K d(x, y) \quad \forall x, y \in M.
\]

The constant \(K\) measures the amount of distortion between \(M\) and its representation \(f(M)\) in Euclidean space. If \(K\) is very close to one, then the mapping \(f\) is almost an embedding and \(f(M)\) can be thought of as a close approximation of \(M\). Thus, it is an interesting question to determine for a given finite metric space \(M\), the smallest constant \(K \geq 1\) for which such a \(K\)-embedding \(f: M \rightarrow \mathbb{R}^n\) exists as above. The answer to this question depends in many cases on the dimension \(n\) of the target space \(\mathbb{R}^n\).

The aim of this bachelor thesis is to investigate questions of \(K\)-embeddability of finite metric spaces into normed spaces. A particular topic of interest will be the famous Johnson-Lindenstrauss Flattening Lemma, which asserts that it is possible to embed ‘truly \(n\)-dimensional’ finite metric spaces (such as the vertex set of the unit cube in \(\mathbb{R}^n\)) into Euclidean spaces of much smaller dimension with distortion arbitrarily close to one.