Between Lipschitz and bilipschitz mappings on Euclidean spaces.

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Key Definitions and Notation

A mapping $f\colon A \subseteq \mathbb{R}^d \to \mathbb{R}^d$ is called

- *Lipschitz* if there exists a constant $L \geq 0$ such that

  $$\|f(y) - f(x)\| \leq L\|y - x\|.$$

  for all $x, y \in A$.
- *bilipschitz* if there exist constants $L \geq b > 0$ such that

  $$b\|y - x\| \leq \|f(y) - f(x)\| \leq L\|y - x\|$$

  for all $x, y \in A$.

We will use the terms L-Lipschitz and (b, L)-bilipschitz in the obvious way.

Between Lipschitz and bilipschitz mappings.

We call a mapping $f\colon A \subseteq \mathbb{R}^d \to \mathbb{R}^d$ *regular* if it satisfies one of the following equivalent conditions:

1. f is Lipschitz and there exists a constant $C > 0$ such that the preimage $f^{-1}(B)$ of any ball $B \subseteq \mathbb{R}^d$ can be covered by C balls of the same radius as B.
2. f is Lipschitz and conserves measure in the sense that

$$\mathcal{L}(f(E)) \geq a\mathcal{L}(E)$$

   for some constant $a > 0$ and any measurable set $E \subseteq A$.

We will call a Lipschitz mapping (C, L)-regular if it is L-Lipschitz and regular with constant C in the sense of 1.

Simple properties of regular mappings.

Let $f\colon A \subseteq \mathbb{R}^d \to \mathbb{R}^d$ be a (C, L)-regular mapping.

- For any point $y \in \mathbb{R}^d$ there are at most C pre-images of y, i.e. $\left|f^{-1}(\{y\})\right| \leq C$.
- A set $E \subseteq A$ has measure zero if and only if $f(E)$ has measure zero.
- Whenever the derivative $Df(x)$ of f at a point $x \in A$ exists, we have that $Df(x)$ is invertible.

A bilipschitz decomposition of Lipschitz mappings.

Proposition (Jones, 1988)

Let $f\colon [0,1]^d \to \mathbb{R}^d$ be a Lipschitz mapping. Then there exist sets $(A_n)_{n=1}^{\infty}$ in $[0,1]^d$ such that

1. *$f|_{A_n}$ is bilipschitz for each n.*
2. *The set $\bigcup_{n=1}^{\infty} f(A_n)$ has full measure in the image of f.*

Jones (1988) proves a quantative version of this statement.

A bilipschitz decomposition of regular mappings

Theorem (D., Kaluža, 2016)

Let $f\colon [0,1]^d \to \mathbb{R}^d$ be a (C,L)-regular mapping. Then there exist open sets $(A_n)_{n=1}^{\infty}$ in $[0,1]^d$ such that

1. $f|_{A_n}$ *is $(b(C),L)$-bilipschitz for each n.*
2. *The set $\bigcup_{n=1}^{\infty} A_n$ is dense (and open) in $[0,1]^d$.*

Structure of the proof

Lemma (D., Kaluža, 2016)

Let $U \subseteq \mathbb{R}^d$ be a non-empty, open set and $f \colon U \to \mathbb{R}^d$ be a (C, L)-regular mapping. Then there exists a non-empty open set $V \subseteq U$ such that

1. $f|_V$ *is almost everywhere injective.*
2. $f|_V$ *is injective.*
3. $f|_V$ *is $(b(C), L)$-bilipschitz.*

A useful lemma.

Lemma (D., Kaluža, 2016)

Let $f\colon I^d \to \mathbb{R}^d$ *be a* (C, L)*-regular mapping. Then there exist* $N \in \{1, 2, \ldots, C\}$*, a non-empty, open set* $T \subseteq f(I^d)$ *and open sets* $W_1, W_2, \ldots, W_N \subseteq I^d$ *such that*

1. *For each* $i \in [N]$ *the mapping* $f\colon W_i \to T$ *is a* $(b(C), L)$*-bilipschitz homeomorphism.*
2. $f^{-1}(T) = \bigcup_{i=1}^{N} W_i$.

Thank you for your attention!