σ-porosity of strict contractions in a space of non-expansive mappings.[1]

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[1] Joint work with Christian Bargetz.

A space of non-expansive mappings.

- Let C be a non-trivial, convex, closed and bounded subset of a Banach space X.
- A mapping $f : C \to C$ is called *non-expansive* if

$$\|f(y) - f(x)\| \leq \|y - x\| \qquad \forall x, y \in C.$$

- We let $\mathcal{M}$ denote the set of all non-expansive mappings $f : C \to C$ and equip $\mathcal{M}$ with the metric

$$d(f, g) = \|f - g\|_{\infty} = \sup \{\|f(x) - g(x)\| : x \in C\}.$$

The subset of strict contractions.

- For a non-expansive mapping $f : C \to C$, we define the Lipschitz constant of f by

$$\operatorname{Lip}(f) := \sup\left\{\frac{\|f(y) - f(x)\|}{\|y - x\|} : x, y \in C,\ y \neq x\right\}.$$

- A non-expansive mapping $f \in \mathcal{M}$ is called a *strict contraction* if $\operatorname{Lip}(f) < 1$.
- We denote by $\mathcal{N}$ the subset of $\mathcal{M}$ formed by all strict contractions.

How big is the set $\mathcal{N}$ of strict contractions in $(\mathcal{M}, d)$?

Some first observations:

- The set $\mathcal{N}$ contains a bi-lipschitz copy of $\mathcal{M}$.
- On the other hand, for each $\lambda \in [0, 1)$, the condition

$$\|f(y) - f(x)\| \leq \lambda \|y - x\| \qquad \forall x, y \in C$$

looks harder to satisfy, than it does to break.

Exceptional sets in complete metric spaces.

Let (S, ρ) be a complete metric space.

Definition

- A subset E of S is called *nowhere dense* if for every point $x \in E$ there exists a sequence of open balls $(B_i)_{i=1}^{\infty}$ in S with midpoints converging to x and $B_i \cap E = \emptyset$ for all i.
- A subset F of S is called *meagre* if F can be expressed as a countable union of nowhere dense sets.
- Moreover, a subset R of S is called *residual* if it is the complement of a meagre set.

Exceptional sets in complete metric spaces.

Let (S, ρ) be a complete metric space.

Definition

- A subset E of S is called *porous* if there exists $c \in (0,1)$ such that for every point $x \in E$ there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there exists $h \in S$ such that $\rho(h, x) \leq \varepsilon$ and $B(h, c\varepsilon) \cap E = \emptyset$.
- A subset F of S is called σ*-porous* if F can be expressed as a countable union of porous sets.

Negligibility of the strict contractions.

The Hilbert space setting.

Let C be a non-trivial, convex, closed and bounded subset of a **Hilbert space** X. Let $(\mathcal{M}, d)$ denote the complete metric space of non-expansive mappings $f : C \to C$ and let $\mathcal{N}$ denote the subset of $\mathcal{M}$ formed by the strict contractions.

Theorem (de Blasi, Myjak 1989)

$\mathcal{N}$ is a σ-porous subset of $\mathcal{M}$.

Negligibility of the set of strict contractions.

Let C be a non-trivial, convex, closed and bounded subset of a **Banach space** X. Let $(\mathcal{M}, d)$ denote the complete metric space of non-expansive mappings $f : C \to C$ and let $\mathcal{N}$ denote the subset of $\mathcal{M}$ formed by the strict contractions.

Theorem (Bargetz, D. 2015)

1. *$\mathcal{N}$ is a σ-porous subset of $\mathcal{M}$.*
2. *If X is separable, then there exists a σ-porous subset $\widetilde{N} \supset \mathcal{N}$ such that every mapping $f \in \mathcal{M} \setminus \widetilde{\mathcal{N}}$ satisfies*

$$\mathrm{Lip}(f, x) = \limsup_{y \to x} \frac{\|f(y) - f(x)\|}{\|y - x\|} = 1$$

at typical points $x \in C$.

A sketch of the proof.

- We may decompose the set $\mathcal{N}$ of strict contractions into countably many slices of the form

$$\mathcal{N}_{a,b} = \{f \in \mathcal{M} \colon a < \mathrm{Lip}(f) \leq b\}$$

  where $a < b$ are real numbers in $(0, 1)$.
- As the distance $b - a$ decreases, the slice $\mathcal{N}_{a,b}$ becomes thinner.
- We prove that sufficiently thin slices of the form $\mathcal{N}_{a,b}$ above are porous.

A sketch of the proof.

An important ingredient in the proof is the following known property of Lipschitz functions on a convex set.

Lemma

Let C be a non-trivial, convex subset of a Banach space X and $f : C \to C$ be a Lipschitz mapping with $\mathrm{Lip}(f) > a > 0$. Then there exist a point $x_0 \in C$ and a direction $e \in S(X)$ such that

$$\liminf_{t \to 0+} \frac{\|f(x_0 + te) - f(x_0)\|}{t} > a.$$

Ongoing research.

- Question: For which complete metric spaces C are the strict contractions on C σ-porous or negligible as a subset of the space of non-expansive mappings on C?
- We can adapt our proof for the Banach space setting to work in all hyperbolic metric spaces.
- Question: Is it possible to strengthen the conclusion that almost all $f \in \mathcal{M}$ satisfy $\mathrm{Lip}(f, x) = 1$ at typical $x \in C$. For example, is it possible that almost all $f \in \mathcal{M}$ satisfy $\mathrm{Lip}(f, x) = 1$ for all $x \in C$?

Thank you for your attention!