A separable Fréchet space of almost universal disposition

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joint work with Jerzy Kąkol and Wiesław Kubiś

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Notation

Let E and F be Banach spaces. A linear mapping

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for all $x \in E$ is called an ε-isometric embedding.

A universal Banach space

Theorem (Banach-Mazur, 1929)

The space $(\mathcal{C}[0,1], \|\cdot\|_\infty)$ *is universal for all separable Banach spaces.*

In other words, for every separable Banach space E there is an isometric embedding $E \hookrightarrow \mathcal{C}[0,1]$.

In 1965, V. I. Gurariĭ constructed a separable Banach space with the following extension property.

(G) For every $\varepsilon > 0$, for all finite dimensional normed spaces $E \subseteq F$, for every isometric embedding $e\colon E \to \mathbb{G}$ there exists an ε-isometric embedding $f\colon F \to \mathbb{G}$ such that $f \restriction E = e$.

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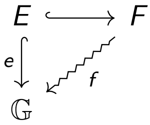

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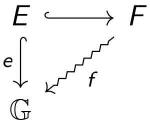

In 1976, W. Lusky showed that (G) defines $\mathbb{G}$ uniquely up to isometry. A simpler proof: Kubiś and Solecki (2013)

Fréchet spaces

A Fréchet space is a complete metrizable locally convex space.

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In terms of semi-norms:
A Fréchet space is a complete topological vector space E with a sequence of semi-norms $\{\|\cdot\|_i\}_{i\in\mathbb{N}}$ such that the sets

$$B_{n,\varepsilon} := \{x \in E \colon \max_{1\leq i\leq n} \|x\|_i < \varepsilon\}$$

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We consider Fréchet spaces with a fixed sequence of semi-norms.

If in addition,

$$\|\cdot\|_1 \le \|\cdot\|_2 \le \dots$$

we call E a graded Fréchet space.

(ε-)Isometries

Let E and F be Fréchet spaces with fixed sequences of semi-norms. A linear mapping

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Theorem (Mazur-Orlicz, 1948)

The space $(\mathcal{C}(\mathbb{R}), \{\|\cdot\|_i\}_{i\in\mathbb{N}})$, *where*

$$\|f\|_i := \sup\left\{|f(x)|\colon x \in [-i, i]\right\},$$

is universal for all separable Fréchet spaces.

In other words, for every separable Fréchet space E there is an isometric embedding $E \hookrightarrow \mathcal{C}(\mathbb{R})$.

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Is there an analogue of the Gurariĭ space for separable Fréchet spaces?

A natural candidate is $\mathbb{G}^{\mathbb{N}}$. Can we find a suitable sequence of semi-norms on $\mathbb{G}^{\mathbb{N}}$?

An important tool: A universal operator on $\mathbb{G}$

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2. $\forall \varepsilon > 0$, $E \subseteq F$ finite dimensional Banach spaces, $\|T\| \leq 1$,

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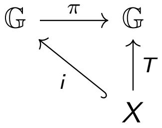

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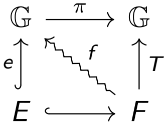

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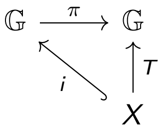

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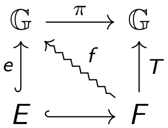

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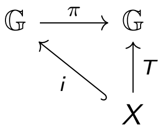

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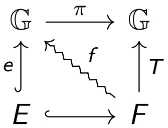

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Based on a result by Cabello Sánchez, Garbulińska-Węgrzyn, Kubiś (2014).

A graded sequence of semi-norms

We construct inductively a sequence of semi-norms on $\mathbb{G}^{\mathbb{N}}$.
For $x = (x_1, x_2, \ldots) \in \mathbb{G}^{\mathbb{N}}$ we define

$$\|x\|_1 := \|x_1\|_{\mathbb{G}}.$$

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From the properties of π, we may conclude that

$$\mathbb{G} = (\operatorname{im} \pi) \oplus (\ker \pi) \simeq \mathbb{G} \oplus \mathbb{G} = \mathbb{G} \times \mathbb{G}$$

holds isometrically and that there is a norm $\| \cdot \|_2'$ on $\mathbb{G}^2$ such that $\mathbb{G}^2$ is isometric to $\mathbb{G}$ and

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Define $\|x\|_2 := \|(x_1, x_2)\|_2'$ and inductively $\|\cdot\|_n$ in a similar way.

$\mathbb{G}^{\mathbb{N}}$ is of almost universal disposition

Proposition

The space $(\mathbb{G}^{\mathbb{N}}, \{\|\cdot\|_i\}_{i\in\mathbb{N}})$ *is a graded Fréchet space of almost universal disposition for finite dimensional graded Fréchet spaces, i.e., for all* $\varepsilon > 0$ *and for all finite dimensional graded Fréchet spaces* $E \subseteq F$ *and all isometric embeddings* $f\colon E \to \mathbb{G}^{\mathbb{N}}$ *there is an* ε*-isometric embedding* $g\colon F \to \mathbb{G}^{\mathbb{N}}$ *such that* $g \restriction E = f$.

Proof sketch

Given $\varepsilon > 0$, finite dimensional graded Fréchet spaces E and F and an isomeric embedding $f\colon E \to F$, we use the notation

$$E_i := (E/\ker\|\cdot\|_{E,i}, \|\cdot\|_{E,i}) \quad \text{and} \quad F_i := (F/\ker\|\cdot\|_{F,i}, \|\cdot\|_{F,i}).$$

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Proof sketch II: Defining g inductively

For $i = 1$, we have

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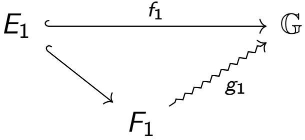

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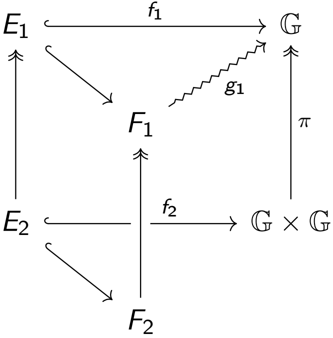

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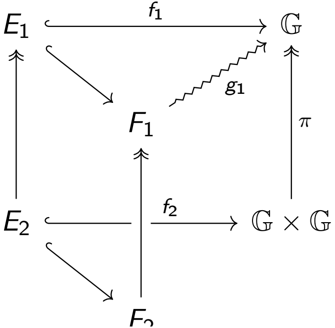

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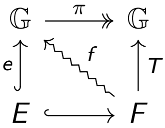

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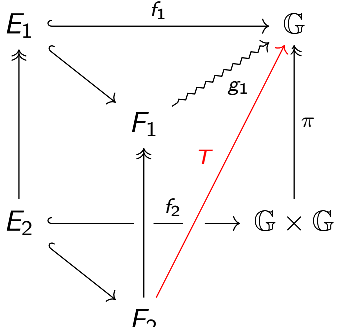

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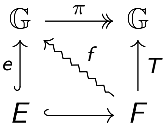

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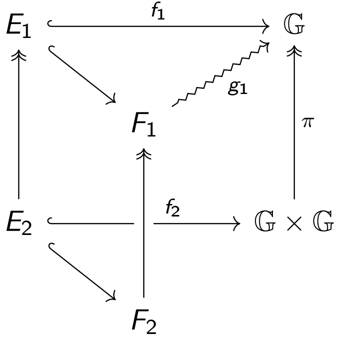

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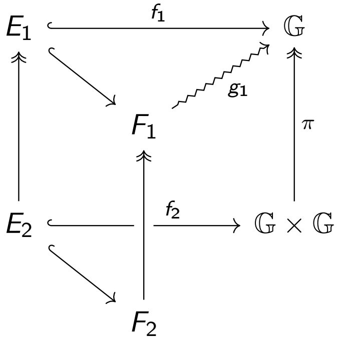

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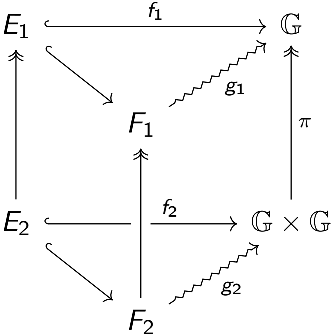

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$$(1+\varepsilon_2)^{-1}(1+\varepsilon_1)^{-1}\|x\|_2 \leq \|g_2(x)\|_2 \leq (1+\varepsilon_2)(1+\varepsilon_1)\|x\|_2.$$

Proof sketch II: Defining g inductively

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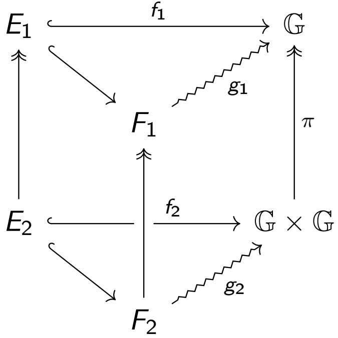

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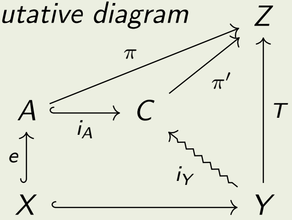
Then we continue inductively.

Lemma

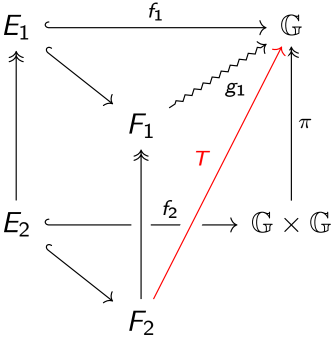
Let $X \subset Y$ and A be finite dim. Banach spaces, Z a Banach space, $e\colon X \hookrightarrow A$, $T\colon Y \to Z$ with $\|T\| < r$, $r > 1$, and $\pi\colon A \to Z$, $\|\pi\| \leq 1$, s.t.

$$\begin{array}{ccc} A & \xrightarrow{\ \pi\ } & Z \\ {\scriptstyle e}\big\uparrow & & \big\uparrow{\scriptstyle T} \\ X & \hookrightarrow & Y \end{array}$$

There exists a finite dim. Banach space C, $i_A\colon A \hookrightarrow C$, an $(r-1)$-isometric embedding $i_Y\colon Y \to C$ and $\pi'\colon C \to Z$, $\|\pi'\| \leq 1$ s.t. we get the commutative diagram



Proof sketch III: The additional step



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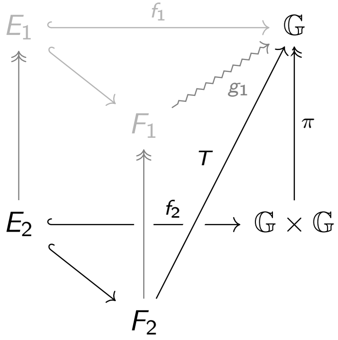

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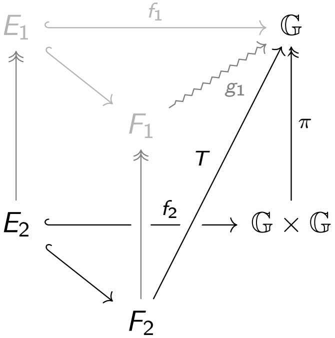

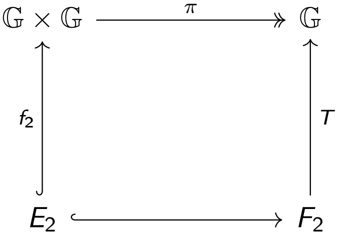

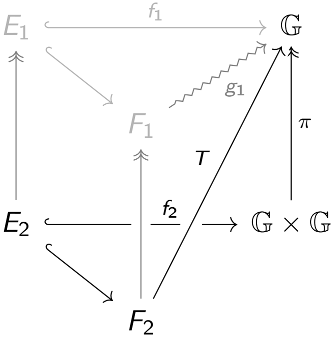
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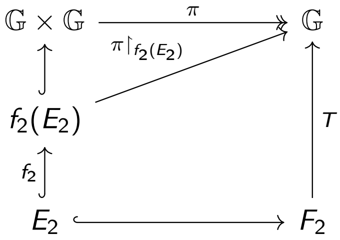
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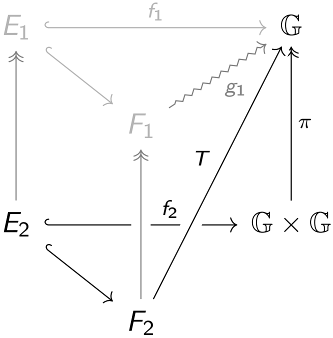
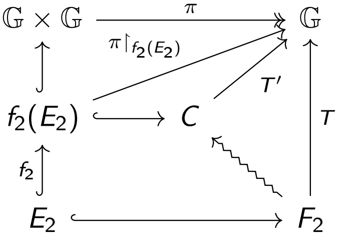
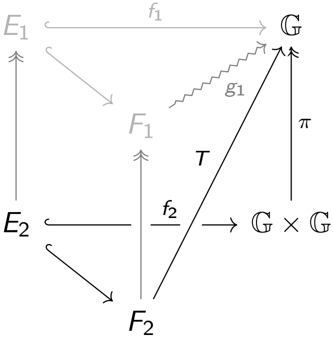




$$\|T'\| \leq 1$$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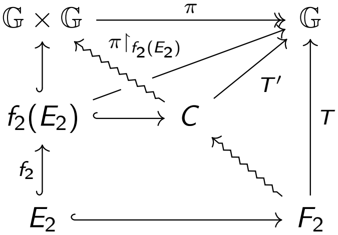
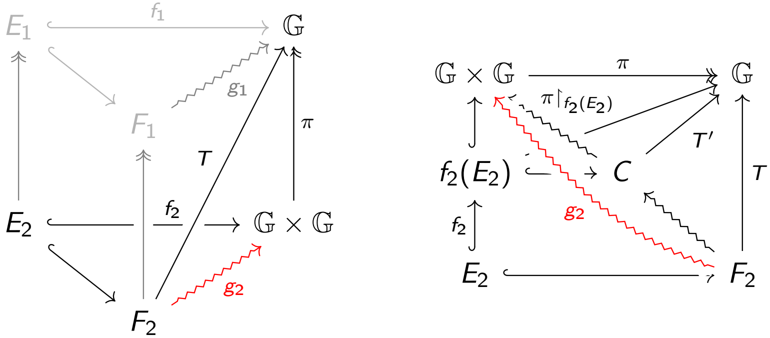
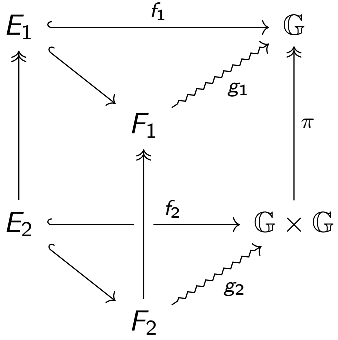
$\mathbb{G} \times \mathbb{G}$
F_2


$\mathbb{G} \times \mathbb{G}$
π
$\mathbb{G}$
$\pi\restriction_{f_2(E_2)}$
T'
$f_2(E_2)$
C
T
f_2
E_2
F_2


E1
f1
𝔾
g1
F1
T
π
E2
f2
𝔾 × 𝔾
F2
g2
𝔾 × 𝔾
π
𝔾
π↾f2(E2)
T′
f2(E2)
C
T
f2
g2
E2
F2

Proof sketch III: The additional step



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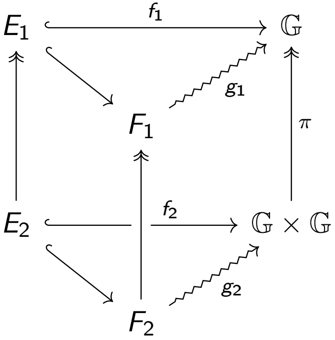

Some Remarks

- If we equip $\mathbb{G}^{\mathbb{N}}$ with the sequence of semi-norms given by

$$\|x\|_i := \|x_i\|_{\mathbb{G}},$$

we get a separable Fréchet space which is of almost universal disposition for finite dimensional Fréchet spaces with fixed (but not necessarily increasing) semi-norms.

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  we get a separable Fréchet space which is of almost universal disposition for finite dimensional Fréchet spaces with fixed (but not necessarily increasing) semi-norms.
- In this case of "independent" semi-norms, the construction is less complicated, since for g_{i+1} it is not necessary to extend g_i.

- The space $\mathbb{G}^{\mathbb{N}}$ is unique up to isometries.

Properties of $\mathbb{G}^{\mathbb{N}}$ / Comparison to the Banach space case

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- No space of the form $\mathcal{C}(X)$, where X is a hemi-compact topological space, is of almost universal disposition for finite dimensional (graded) Fréchet spaces.

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