A sequence space representation of L. Schwartz' space $\mathcal{O}_C$

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Preliminaries: Topological Tensor Products

Let E and F be two separated locally convex spaces. We use the following two topologies on the tensor product $E \otimes F$.

$E \otimes_\pi F$ . . . finest locally convex topology such that

$$\mathrm{can}\colon E \times F \to E \otimes F$$

is continuous.

$E \otimes_\iota F$ . . . finest locally convex topology such that

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$E \widehat{\otimes}_\pi F$ and $E \widehat{\otimes}_\iota F$ completion of $E \otimes_\pi F$ and $E \otimes_\iota F$, respectively.

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The Valdivia-Vogt structure table

$$\mathcal{D} \subset \mathcal{S} \subset \mathcal{D}_{L^p} \subset \dot{\mathcal{B}} \subset \mathcal{D}_{L^\infty} \subset \mathcal{O}_C \subset \mathcal{O}_M \subset \mathcal{E}$$

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$$\mathcal{D} = \mathcal{D}(\mathbb{R}^n) = \{f \in \mathcal{C}^\infty(\mathbb{R}^n); \operatorname{supp} f \text{ compact}\}$$

$$s^{(\mathbb{N})} = \left\{(x(j))_{j\in\mathbb{N}} \in s^{\mathbb{N}}; \exists N \, \forall k \geq N \colon x(k) = 0\right\}$$

$$\cong \lim_{k\to} (s)^k = \lim_{k\to} (\mathbb{C}^k \widehat{\otimes} s) = \mathbb{C}^{(\mathbb{N})} \widehat{\otimes}_\iota s$$

Representation: M. Valdivia (1978), D. Vogt (1983), Ch.B. (2012)

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$$\mathcal{S} = \mathcal{S}(\mathbb{R}^n) = \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^n); \forall k \in \mathbb{N}_0 \forall \alpha \in \mathbb{N}_0^n \colon (1+|x|^2)^{k/2} \partial^\alpha f(x) \in \mathcal{C}_0 \right\}$$

$$s\widehat{\otimes}s = \{(x(i,j))_{(i,j)\in\mathbb{N}^2}; \forall k \in \mathbb{N}_0 \forall l \in \mathbb{N}_0 \colon \sup_{i,j\in\mathbb{N}} |i^k j^l x(i,j)| < \infty\}$$

Representation: A. Grothendieck (1955), L. Schwartz (1957), M. Valdivia (1982) and R. Meise and D. Vogt (1992)

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$$\ell^p\widehat{\otimes}s = \left\{(x(i,j))_{(i,j)\in\mathbb{N}^2}; \forall k \in \mathbb{N}_0\colon \sup_{j\in\mathbb{N}} \left(\sum_{i\in\mathbb{N}} j^{pk}|x(i,j)|^p\right)^{1/p} < \infty\right\}$$

Representation: M. Valdivia (1981), D. Vogt (1983), N. Ortner and P. Wagner (2011)

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s^{(\mathbb{N})} & & s\widehat{\otimes}s & \subset & \ell^p\widehat{\otimes}s & \subset & c_0\widehat{\otimes}s & & & & & & & &
\end{array}$$

$$\dot{\mathcal{B}} = \dot{\mathcal{B}}(\mathbb{R}^n) = \{f \in \mathcal{C}^\infty(\mathbb{R}^n); \forall \alpha \in \mathbb{N}_0^n \colon \partial^\alpha f \in \mathcal{C}_0(\mathbb{R}^n)\}$$

$$c_0\widehat{\otimes}s = \{(x(i,j))_{(i,j)\in\mathbb{N}^2}; \forall k \in \mathbb{N} \lim_{i\to\infty} \sup_{j\in\mathbb{N}} |j^k x(i,j)| = 0\}$$

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$$\ell^\infty\widehat{\otimes}s = \{(x(i,j))_{(i,j)\in\mathbb{N}^2}; \forall k \in \mathbb{N}_0\colon \sup_{i,j\in\mathbb{N}} \{|j^k x(i,j)|\} < \infty\}$$

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\end{array}$$

$$\mathcal{O}_M = \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^n); \forall \alpha \in \mathbb{N}_0^n \, \exists k \in \mathbb{N}_0 \colon \sup_{x\in\mathbb{R}^n} (1+|x|^2)^{-\frac{k}{2}} |\partial^\alpha f(x)| < \infty \right\}$$

$$s'\widehat{\otimes}_\pi s = \{(x(i,j))_{(i,j)\in\mathbb{N}^2}; \forall l \in \mathbb{N}_0 \exists k \in \mathbb{N}_0 \colon \sup_{i,j\in\mathbb{N}} |i^l j^{-k} x(i,j)| < \infty\}$$

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Representation: Topic of this talk

Two results of A. Grothendieck

Proposition (Grothendieck thèse, Chap. II, p. 90)

Let E and F be complete locally convex spaces, E nuclear and F semi-reflexive. It holds

$$(E \widehat{\otimes}_\pi F)' = E' \widehat{\otimes}_\iota F'$$

if and only if $(E \widehat{\otimes}_\pi F)'$ is complete.

Proposition (Grothendieck thèse, Chap. II, p. 128)

The space $s' \widehat{\otimes}_\pi s$ is bornological.

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A representation as a tensor product of sequence spaces

Proposition

The spaces $\mathcal{O}_C$ and $\mathcal{O}'_M$ have the sequence space representation

$$\mathcal{O}_C \cong \mathcal{O}'_M \cong s' \widehat{\otimes}_\iota s.$$

Proof.

By Fourier transform we get $\mathcal{O}_C \cong \mathcal{O}'_M$.
M. Valdivia: $\mathcal{O}_M \cong s' \widehat{\otimes}_\pi s$. Hence $\mathcal{O}'_M \cong (s' \widehat{\otimes}_\pi s)'$.
$s' \widehat{\otimes}_\pi s$ is bornological $\Rightarrow (s' \widehat{\otimes}_\pi s)'$ complete
Therefore $(s' \widehat{\otimes}_\pi s)' = s \widehat{\otimes}_\iota s'$, as s and s' are complete, nuclear and reflexive. □

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A representation of $s'\widehat{\otimes}_\iota s$ as an inductive limit

Proposition

We have the limit representation

$$s'\widehat{\otimes}_\iota s = \lim_{k\to}((\ell^\infty)_{-k}\widehat{\otimes}s) = \lim_{k\to}\lim_{\leftarrow l}\ell^\infty((\ell^\infty)_{-k})_l.$$

In particular it holds

$$s'\widehat{\otimes}_\iota s = \left\{(x(i,j))_{i,j\in\mathbb{N}};\, \exists k \in \mathbb{N}_0\, \forall l \in \mathbb{N}_0 \colon \sup_{i,j\in\mathbb{N}} |i^{-k}j^l x(i,j)| < \infty\right\}.$$

Sketch of Proof.

$s\widehat{\otimes}_\pi s' = \lim_{\leftarrow k}((\ell^1)_k\widehat{\otimes}s')$

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A decomposition result

Proposition (L. Schwartz, 1957)

The identities

$$\mathcal{E}_{x,y} = \mathcal{E}_x \widehat{\otimes} \mathcal{E}_y \qquad \mathcal{S}_{x,y} = \mathcal{S}_x \widehat{\otimes} \mathcal{S}_y \qquad (\mathcal{O}_M)_{x,y} = (\mathcal{O}_M)_x \widehat{\otimes}_\pi (\mathcal{O}_M)_y$$

and

$$\mathcal{D}_{x,y} = \mathcal{D}_x \widehat{\otimes}_\iota \mathcal{D}_y$$

hold.

Is there an equivalent result for $(\mathcal{O}_C)_{x,y}$?

Proposition

The identity

$$(\mathcal{O}_C)_{x,y} = (\mathcal{O}_C)_x \widehat{\otimes}_\iota (\mathcal{O}_C)_y$$

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holds

Proof.

Fourier transform: $\mathcal{O}'_C \cong \mathcal{O}_M$.
Above result (L. Schwartz):

$$(\mathcal{O}'_C)_{x,y} \cong (\mathcal{O}_M)_{x,y} = (\mathcal{O}_M)_x \widehat{\otimes}_\pi (\mathcal{O}_M)_y \cong (\mathcal{O}'_C)_x \widehat{\otimes}_\pi (\mathcal{O}'_C)_y$$

Hence:

$$(\mathcal{O}_C)_{x,y} = ((\mathcal{O}'_C)_{x,y})' = ((\mathcal{O}'_C)_x \widehat{\otimes}_\pi (\mathcal{O}'_C)_y)' = (\mathcal{O}_C)_x \widehat{\otimes}_\iota (\mathcal{O}_C)_y$$

as $\mathcal{O}_C$ is bornological, reflexive and nuclear. □

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