# Impulsive Solutions of Differential Behaviors

diploma thesis in mathematics

by

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# Contents

# 1 Introduction

This master's thesis is based on the article [1] of H. Bourlès on impulsive systems but presents this theory from a behavioral point of view as suggested by U. Oberst. We consider signal spaces $\mathscr{F}$ of univariate functions or distributions on which the complex polynomial algebra $\mathbb{C}[s]$ in one indeterminate $s$ acts by differentiation, i.e., by $s \circ y = \frac{\mathrm{d}y}{\mathrm{d}t}$. The behaviors are solution modules of the form

$$\mathcal{B} = \left\{ w \in \mathcal{F}^l; R \circ w = 0 \right\}$$

where $R \in \mathbb{C}[s]^{k \times l}$ is a polynomial matrix. In particular IO(input/output) behaviors are of the form

$$\mathcal{B} = \left\{ \begin{pmatrix} y \\ u \end{pmatrix} \in \mathcal{F}^{p+m}; P \circ y = Q \circ u \right\}$$

with $P \in \mathbb{C}[s]^{k \times p}$, $Q \in \mathbb{C}[s]^{k \times m}$, $PH = Q$ and $\operatorname{rank}(P, -Q) = \operatorname{rank}(P) = p$. The matrix $H$ is the transfer matrix of the system. The concept of impulsive behavior was introduced by G. C. Verghese and T. Kailath in [4] and [5]. A trajectory $\binom{y}{u}$ of the IO behavior is called impulsive if its input $u$ is piecewise smooth, but its output $y$ is not piecewise continuous. As usual we admit piecewise smooth inputs of the form $u = u_1 + u_2 Y$ only where $Y(t)$ is the Heaviside or unit step function and where $u_1$ and $u_2$ are smooth. The output $y$ is of the form

$$y = y_1 + y_2 Y + y_3,\ y_3 \in \oplus_{n=0}^{\infty} \mathbb{C}^p \delta^{(n)}$$

where again $y_1$ and $y_2$ are smooth, but the impulsive component $y_3$ is a linear combination of derivatives $\delta^{(n)}$ of the $\delta$- distribution $\delta = Y'$ at 0. The impulsive components $y_3$ of $\mathcal{B}$ do not themselves form a behavior. However, under the canonical isomorphism

$$\mathbb{C}[s] \circ \delta = \oplus_{n=0}^{\infty} \mathbb{C}\delta^{(n)} \cong \mathbb{C}^{(\mathbb{N})}, \sum a_n \delta^{(n)} \mapsto (a_0, a_1, a_2, \ldots)$$

the set of impulsive components of $\mathcal{B}$ maps onto a discrete behavior $\mathcal{B}_\infty \subseteq (\mathbb{C}^{(\mathbb{N})})^p$ where the action on $\mathbb{C}^{(\mathbb{N})}$ is the left shift action. This is the impulsive behavior associated with $\mathcal{B}$ according to Bourlès. It is autonomous and thus $\mathbb{C}$-finite-dimensional. For computational purposes we assume additionally that the functions $u_1$ and $u_2$ and then also $y_1$ and $y_2$ are polynomial-exponential since these functions can be described by finitely many data.
We give a criterion to decide whether a given IO system contains impulsive solutions and develop an algorithm to compute the impulsive behavior of a given transfer matrix and a formula for the impulsive part of the output, corresponding to a given input. We discuss different signal spaces that are either injective cogenerators or vector spaces over the quotient field $\mathbb{C}(s)$ of the complex polynomial ring $\mathbb{C}[s]$.

# 2 Abstract Linear System Theory

This section is based on the lecture "Eindimensionale lineare Systemtheorie" by Professor Ulrich Oberst in the winter term 2004/05. We introduce the necessary fundamental concepts of linear system theory from a behavioral point of view. In 2.1 we consider the duality of modules and behaviors and introduce the concept of controllability. In 2.2 we consider transfer spaces of behaviors. In 2.3 we define input-output systems and introduce the transfer matrix of an input-output system and the unique controllable realisation of a given transfer matrix.

## 2.1 Foundations

**Definition 1.** Let $V$ be a $\mathbb{C}[s]$-module.

1. $V$ is *divisible* if and only if for all nonzero polynomials $0 \neq f \in \mathbb{C}[s]$ the multiplication $f \circ (-) : V \to V$ is surjective. Since over $\mathbb{C}$ all polynomials split into linear factors $f = a\prod_{\lambda \in V_{\mathbb{C}}(f)}(s-\lambda)^{\mu(\lambda)}$, divisibility signifies that for all complex numbers $\lambda \in \mathbb{C}$ the multiplication $(s-\lambda)\circ : V \to V$ is surjective.

2. $V$ is *torsionfree* if and only if for all nonzero polynomials $0 \neq f \in \mathbb{C}[s]$ the multiplication $f \circ (-) : V \to V$ is injective, i.e., if for all complex numbers $\lambda \in \mathbb{C}$ the multiplication $(s-\lambda)\circ : V \to V$ is injective.

**Remark 1.** $V$ is divisible and torsionfree if and only if $V$ is a $\mathbb{C}(s)$-vector space over the quotient field $\mathbb{C}(s)$ of $\mathbb{C}[s]$.

In the sequel let $\mathscr{D}$ denote a commutative, noetherian integral domain and $\mathscr{F}$ a $\mathscr{D}$-module with scalar multiplication $d \circ y$, $d \in \mathscr{D}$, $y \in \mathscr{F}$.

**Definition 2.** We call

$$\mathcal{B} := \left\{ y \in \mathscr{F}^l ; R \circ y = 0 \right\} \subset \mathscr{F}^l$$

the behavior associated with the matrix $R$. It is a $\mathscr{D}$-submodule of $\mathscr{F}^l$.

**Reminder 1.** $\mathrm{mod}_{\mathscr{D}}$ is the category of $\mathscr{D}$-modules. The duality functor $D := \mathrm{Hom}_{\mathscr{D}}(-, \mathscr{F})$ is contravariant, i.e.,

$$\begin{array}{ccc} D : \mathrm{mod}_{\mathscr{D}}^{op} & \longrightarrow & \mathrm{mod}_{\mathscr{D}} \\ M_1 & & DM_1 = \mathrm{Hom}_{\mathscr{D}}(M_1, \mathscr{F}) \\ {\scriptstyle f}\downarrow & \longmapsto & \uparrow {\scriptstyle Df = \mathrm{Hom}_{\mathscr{D}}(f, \mathscr{F})} \\ M_2 & & DM_2 = \mathrm{Hom}_{\mathscr{D}}(M_2, \mathscr{F}) \end{array}$$

and left exact, i.e, if the sequence

$$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$

is exact then the sequence

$$DM_1 \xleftarrow{Df} DM_2 \xleftarrow{Dg} DM_3 \longleftarrow 0$$

is exact too.

We consider $\mathscr{D}^{1\times l}$ with the standard basis $\delta_i$. Every element $\xi \in \mathscr{D}^{1\times l}$ is of the form

$$\xi = \sum_{j=1}^{l} \xi_j \delta_j .$$

We consider the isomorphism

$$\mathrm{Hom}_{\mathscr{D}}(\mathscr{D}^{1\times l}, \mathscr{F}) \cong \mathscr{F}^l, \varphi \leftrightarrow w = (w_j)_j,$$
$$\varphi(\delta_j) = w_j, \varphi(\xi) = \xi \circ w = \sum_{j=1}^{l} \xi_j w_j .$$

From linear algebra we know that the $\mathscr{D}$-module of the $k \times l$ matrices $\mathscr{D}^{k\times l}$ is isomorphic to the module of the linear maps from $\mathscr{D}^{1\times k}$ to $\mathscr{D}^{1\times l}$, i.e.,

$$\mathrm{Hom}_{\mathscr{D}}(\mathscr{D}^{1\times k}, \mathscr{D}^{1\times l}) \cong \mathscr{D}^{k\times l}, (\circ E) \leftrightarrow E.$$

**Lemma 1.** *Let $\circ E : \mathscr{D}^{1\times k} \to \mathscr{D}^{1\times l}$ be a linear map. Then the diagram*

$$\begin{tikzcd}
D(\circ E) : \mathrm{Hom}_{\mathscr{D}}(\mathscr{D}^{1\times k}, \mathscr{F}) \arrow[r, "D(\circ E)"] & \mathrm{Hom}_{\mathscr{D}}(\mathscr{D}^{1\times l}, \mathscr{F}) \\
\mathscr{F}^l \arrow[u, leftrightarrow, "\cong"] \arrow[r, "E\circ"] & \mathscr{F}^k \arrow[u, leftrightarrow, "\cong"]
\end{tikzcd}$$

*is commutative.*

*Proof.* In the diagram above we consider

$$\begin{tikzcd}
\varphi \arrow[r, mapsto, "D(\circ E)"] \arrow[d, leftrightarrow] & \varphi(\circ E) \arrow[d, leftrightarrow] \\
y \arrow[r, mapsto, "E\circ"] & E \circ y \overset{?}{=} u
\end{tikzcd}$$

where

$$\begin{aligned}
u_i &= (\varphi(\circ E))\,(\delta_i) = \varphi(\delta_i E) = \varphi(E_{i-}) = \varphi(\sum_{j=1}^{l} E_{ij}\delta_j) = \sum_{j=1}^{l} E_{ij} \circ \varphi(\delta_j) \\
&= \sum_{j=1}^{l} E_{ij} \circ y_i = (E \circ y)_i
\end{aligned}$$

□

**Reminder 2.** Let $\mathscr{F}$ be a $\mathscr{D}$-module. $\mathscr{F}$ is injective $\Leftrightarrow D = \mathrm{Hom}_{\mathscr{D}}(-, \mathscr{F})$ is exact or, equivalently, if for all $\mathscr{D}$-modules $N$ and for all submodules $M \subseteq N$ the map

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{D}}(N, \mathscr{F}) \arrow[r, "\mathrm{Hom}_{\mathscr{D}}(\mathrm{inj}, \mathscr{F})"] & \mathrm{Hom}_{\mathscr{D}}(M, \mathscr{F}) \\
g \arrow[r, mapsto, dashed] \arrow[dr, mapsto] & f \arrow[d, equal] \\
& \mathrm{Hom}_{\mathscr{D}}(\mathrm{inj}, \mathscr{F})(g) = g\ \mathrm{inj} = g|_M
\end{tikzcd}$$

is surjective, i.e., if every $f : M \to \mathscr{F}$ has an extension to $g : N \to \mathscr{F}$.

**Theorem and Definition 1.** *Let $\mathscr{F}$ be an injective $\mathscr{D}$-module. The following properties are equivalent*

1. *For all nonzero $f \neq 0$ the dual map $Df = \mathrm{Hom}(f, \mathscr{F}) \neq 0$ is nonzero.*
2. *For all nonzero modules $M \neq 0$ the dual module $DM = \mathrm{Hom}(M, \mathscr{F}) \neq 0$ is nonzero too.*

*If these properties are satisfied $\mathscr{F}$ is called an* injective cogenerator.

*Proof.* $\Rightarrow$: Let $M$ be a $\mathscr{D}$-module, it is nonzero if and only if the identity map $\mathrm{id}_M \neq 0$ is nonzero. Therefore

$$0 \neq D\mathrm{id}_M = \mathrm{id}_{DM} \Leftrightarrow DM \neq 0.$$

$\Leftarrow$: Let $f : M_1 \to M_2$ a $\mathscr{D}$-linear map. We define $M_3 := \mathrm{im}(f)$ it is nonzero if and only if $f \neq 0$, i.e., $0 \neq f \Leftrightarrow M_3 \neq 0$. We assume that $f \neq 0$ and therefore $M_3 \neq 0$. We consider the sequence

$$M_1 \xrightarrow{f_1 = f_{\mathrm{ind}}} M_3 \xrightarrow{f_2 = \mathrm{inj}} M_2 \ ,$$

where $f = f_2 f_1$. The map $f_1$ is surjective and $f_2$ is injective. $D = \mathrm{Hom}(-, \mathscr{F})$ is exact since $\mathscr{F}$ is injective. Hence $Df_1$ is injective and $Df_2$ is surjective. Since $DM_3 \neq 0$ by assumption and $Df = D(f_2 f_1) = Df_1 Df_2$ we conclude

$$\mathrm{im}(Df) = \mathrm{im}(Df_1)\underbrace{(DM_3)}_{\neq 0} \neq 0.$$

Therefore the dual map $Df \neq 0$ is nonzero.

□

**Example 1.** If F is an arbitrary field and $\mathscr{D} = F[s]$ where $s$ is the left shift the module ${}_{F[s]}F^{\mathbb{N}}$ is an injective cogenerator. If we choose $\mathscr{D} = \mathbb{C}[s]$ where $s$ is the derivative the modules ${}_{\mathbb{C}[s]}\mathcal{C}^\infty$ and ${}_{\mathbb{C}[s]}\mathcal{D}'$ are injective cogenerators.

**Definition 3.** Let ${}_{\mathscr{D}}\mathscr{F}$ be an injective cogenerator. Given a Matrix $R \in \mathscr{D}^{k \times l}$ we consider its row module

$$U := Z(R) = \sum_{i=1}^{k} \mathscr{D} R_{i-} = \mathscr{D}^{1\times k} R = \mathrm{im}(\circ R).$$

Let $\mathcal{B} \subseteq \mathscr{F}^l$ be the behavior associated to $R$. We define the module associated to $\mathcal{B}$ by

$$M(\mathcal{B}) := \mathscr{D}^{1\times l}/U = \mathscr{D}^{1\times l}/\mathscr{D}^{1\times k}R = \mathrm{cok}(\circ R).$$

**Theorem 2** (Malgrange)**.** *Let $R \in \mathscr{D}^{k\times l}$ be a matrix with entries in $\mathscr{D}$ and $U =$ Z$(R)$ its row module. We consider the factor module $M := \mathscr{D}^{1\times l}/U$. Its dual module $DM = \mathrm{Hom}_{\mathscr{D}}(M, \mathscr{F})$ is isomorphic to $\mathcal{B} = \ker(\circ R)$, i.e.,*

$$DM = \mathrm{Hom}_{\mathscr{D}}(M, \mathscr{F}) \cong \mathcal{B} = \left\{ y \in \mathscr{F}^l ; R \circ y = 0 \right\} \subset \mathscr{F}^l, \psi \leftrightarrow y$$

*where*

$$\psi(\bar{\delta}_j) = y_j,\ \psi(\bar{\xi}) = \sum_{j=1}^{l} \xi_j \circ y_j = \xi \circ y$$

*Proof.* The sequence

$$\mathscr{D}^{1\times k} \xrightarrow{\circ R} \mathscr{D}^{1\times l} \xrightarrow{\mathrm{can}} M \longrightarrow 0$$

is exact. Hence the sequences

$$\begin{tikzcd}
0 \arrow[r] & DM = \mathrm{Hom}_{\mathscr{D}}(M, \mathscr{F}) \arrow[r, "D\mathrm{can}"] \arrow[d, leftrightarrow] & \mathrm{Hom}_{\mathscr{D}}(\mathscr{D}^{1\times l}, \mathscr{F}) \arrow[r, "D(\circ R)"] \arrow[d, leftrightarrow, "\cong"] & \mathrm{Hom}_{\mathscr{D}}(\mathscr{D}^{1\times k}, \mathscr{F}) \arrow[d, leftrightarrow, "\cong"] \\
0 \arrow[r] & \mathcal{B} \arrow[r, hook] & \mathscr{F}^l \arrow[r, "R\circ"] & \mathscr{F}^k
\end{tikzcd}$$

are exact too and therefore $DM \cong \mathcal{B}$. □

**Definition 4.** Let $U \subseteq \mathscr{D}^{1\times l}$ be a submodule, we define the *orthogonal submodule* by

$$U^{\perp} := \left\{y \in \mathscr{F}^l; U \circ y = 0\right\} = \left\{y \in \mathscr{F}^l; \forall \xi \in U : \xi \circ y = 0\right\}.$$

Analogously we define for a submodule $\mathcal{B} \subseteq \mathscr{F}^l$

$$\mathcal{B}^{\perp} := \left\{\xi \in \mathscr{D}^{1\times l}; \xi \circ \mathcal{B} = 0\right\} = \left\{\xi \in \mathscr{D}^{1\times l}; \forall y \in \mathcal{B} : \xi \circ y = 0\right\}.$$

Let $\mathcal{B} \subseteq \mathscr{F}^l$ a behavior, i.e.,

$$\mathcal{B} = \left\{y \in \mathscr{F}^l; R \circ y = 0\right\} = U^{\perp}$$

where $U = \mathscr{D}^{1\times k}R$. We define the *associated submodule* of $\mathcal{B}$ by $U(\mathcal{B}) := \mathcal{B}^{\perp}$.

**Example 2.** Let $R \in \mathscr{D}^{k\times l}$, $\mathcal{B}$ the behavior associated to $R$. Let

$$U = \mathscr{D}^{1\times k}R \subseteq \mathscr{D}^{1\times l} \text{ and } M(\mathcal{B}) = \mathscr{D}^{1\times l}/U$$

the module associated to $\mathcal{B}$. Then $\mathcal{B} = U^{\perp}$.

**Theorem 3.** *Let $\mathscr{D}$ be noetherian integral domain and ${}_{\mathscr{D}}\mathscr{F}$ an injective cogenerator. Then*

1. *If $U \subseteq \mathscr{D}^{1\times l}$ is a submodule and $\mathcal{B} := U^{\perp}$ then $U = \mathcal{B}^{\perp}$.*
2. *Let $R_i \in \mathscr{D}^{k(i)\times l}$, $i = 1, 2$ be two matrices, $U_i = \mathscr{D}^{1\times l}R_i$ their row modules and $\mathcal{B}_i = U_i^{\perp} = \left\{y \in \mathscr{F}^l; R_i \circ y = 0\right\}$ their associated behaviors. Then the following holds:*

$$\mathcal{B}_1 \subseteq \mathcal{B}_2 \Leftrightarrow U_2 \subseteq U_1 \Leftrightarrow \exists X : R_2 = XR_1$$

*Proof.*

1. We have to show that $U = \mathcal{B}^{\perp}$.

   $\subseteq$: Let $\xi \in U$ and therefore $\xi \circ y = 0$ for all $y \in \mathcal{B} = U^{\perp}$. Hence $\xi \in U^{\perp\perp} = (U^{\perp})^{\perp} = \mathcal{B}^{\perp}$.

   $\supseteq$: $\mathcal{B}^{\perp} \subseteq U$ is equivalent to $\xi \notin U \Rightarrow \xi \notin \mathcal{B}^{\perp}$. Let $\xi \in \mathscr{D}^{1\times l} \setminus U$. We have to show that $\xi \notin \mathcal{B}^{\perp}$, i.e., $\xi \circ \mathcal{B} \neq 0$. We consider the Gelfand map

   $$\rho_M : M := \mathscr{D}^{1\times l}/U \longrightarrow \mathrm{Hom}_{\mathscr{D}}(\mathrm{Hom}_{\mathscr{D}}(M, \mathscr{F}), \mathscr{F}) \subseteq \mathscr{F}^{DM}$$

   If $\xi \notin U$ the residue class $x = \bar{\xi} = \xi + U \neq 0$ is nonzero. Its image $\rho_M(x) = (f(x))_{f\in DM} \neq 0$ is distinct from zero too since $\mathscr{F}$ is an injective cogenerator. Now we choose an $f$ with nonzero $f(x)$ and consider the Malgrange isomorphism

   $$\mathrm{Hom}(M, \mathscr{F}) \cong \mathcal{B} = \{y; R \circ y = 0\}, f \mapsto \begin{pmatrix} f(\bar{\delta}_1) \\ \vdots \\ f(\bar{\delta}_l) \end{pmatrix} =: y.$$

   Then

   $$0 \neq f(\bar{\xi}) = f(\sum_j \xi_j \bar{\delta}_j) = \sum_j \xi_j \circ \underbrace{f(\bar{\delta}_j)}_{=y_j} = \xi \circ y$$

   is nonzero too and therefore $\xi \notin \mathcal{B}^{\perp}$.

2. We have to show that $\mathcal{B}_1 \subseteq \mathcal{B}_2 \Leftrightarrow U_2 \subseteq U_1$.

$\Leftarrow$: Let $y \in \mathcal{B}_1$, i.e., $\xi \circ y = 0$ for all $\xi \in U_1$. Hence for all $\xi \in U_2 \subseteq U_1$ : $\xi \circ y = 0$ and therefore $y \in \mathcal{B}_2$.

$\Rightarrow$: Let $U_1^\perp = \mathcal{B}_1 \subseteq \mathcal{B}_2 = U_2^\perp$ and hence $\underbrace{U_2^{\perp\perp}}_{=U_2} \subseteq \underbrace{U_1^{\perp\perp}}_{=U_1}$ as above.

$$\begin{aligned} U_2 = \mathscr{D}^{1\times k(2)} R_2 \subseteq U_1 = \mathscr{D}^{1\times k(1)} R_1 \quad &\Leftrightarrow \quad \forall (R_2)_{i-} \exists \eta_i : (R_2)_{i-} = \eta_i R_1 \\ &\Leftrightarrow \quad \exists X = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_{k(2)} \end{pmatrix} : R_2 = X R_1 \end{aligned}$$

$\square$

**Definition 5.** A behavior $\mathcal{B} = \left\{ y \in \mathscr{F}^l ; R \circ y = 0 \right\}$ is said to be *controllable* if its associated module

$$M(\mathcal{B}) = \mathscr{D}^{1\times l} / \mathscr{D}^{1\times k} R$$

is torsionfree.

## 2.2 Transfer Space

Let $\mathscr{D}$ be noetherian integral domain and ${}_{\mathscr{D}}\mathscr{F}$ an injective cogenerator. Let $R \in \mathscr{D}^{k\times l}$ and $\mathcal{B}$ its associated behavior. Let $\mathcal{B}^\perp$ be its associated submodule and $M = M(\mathcal{B}) = \mathscr{D}^{1\times k} / \mathcal{B}^\perp$ its associated module. Let $K = \operatorname{quot}(\mathscr{D})$ the quotient field of $\mathscr{D}$. Let $p = \operatorname{rank}(R)$ and $m = \operatorname{rank}(M) := \dim_K(K \otimes_{\mathscr{D}} M)$ be the ranks of $R$ and $M$ and $l = p + m$.

**Reminder 3.** The tensor product $K \otimes_{\mathscr{D}} M$ is a $K$-vector space with $K$-basis $1 \otimes x_1, \ldots, 1 \otimes x_n$ where $x_1, \ldots, x_n$ is a $\mathscr{D}$-basis of $M$. We identify $K \otimes_{\mathscr{D}} \mathscr{D}^{1\times l} = K^{1\times l}$ and $1 \otimes \delta_i = \delta_i$. Hence

$$\sum_{i=1}^{l} \xi_i \otimes \delta_i = \sum_{i=1}^{l} \xi_i \delta_i = (\xi_1, \ldots, \xi_l) \in K^{1\times l}.$$

Since $K \otimes_{\mathscr{D}} (-) : \operatorname{mod}_{\mathscr{D}} \to \operatorname{mod}_{\mathscr{D}}$ is a left adjoint functor, it is right exact.

**Lemma 2.** $K \otimes_{\mathscr{D}} M \cong K^{1\times l} / K^{1\times k} R$

*Proof.* The sequence

$$\mathscr{D}^{1\times k} \xrightarrow{\circ R} \mathscr{D}^{1\times l} \xrightarrow{\text{can}} M \longrightarrow 0$$

is exact and therefore the sequences

$$\begin{tikzcd} K \otimes \mathscr{D}^{1\times k} \arrow[r, "K\otimes(\circ R)"] \arrow[d, leftrightarrow, "\cong"] & K \otimes \mathscr{D}^{1\times l} \arrow[r, "K\otimes\text{can}"] \arrow[d, leftrightarrow, "\cong"] & K \otimes M \arrow[r] \arrow[d, leftrightarrow] & 0 \arrow[d, equal] \\ K^{1\times k} \arrow[r, "\circ R"] & K^{1\times l} \arrow[r, "\text{can}"] & K^{1\times l}/K^{1\times k}R \arrow[r] & 0 \end{tikzcd}$$

are exact too and therefore $K \otimes_{\mathscr{D}} M \cong K^{1\times l} / K^{1\times k} R$. $\square$

**Lemma 3.** *We consider the canonical map*

$$\mathscr{D}^{1\times l}/\mathscr{D}^{1\times k}R = M \xrightarrow{\text{can}} K \otimes_{\mathscr{D}} M = K^{1\times l}/K^{1\times k}R$$

*The kernel of* can *is the torsion submodule of* $M$*, i.e.,*

$$\ker(\text{can}) = \left(\mathscr{D}^{1\times l} \cap K^{1\times k}R\right)/\mathscr{D}^{1\times k}R = \mathrm{T}(M).$$

*Proof.* $\subseteq$: Let $\bar{\xi} \in \ker(\text{can})$ hence $\bar{\xi} = 0$ in $K^{1\times l}/K^{1\times k}R$. Consequently $\xi$ is a $K$-linear combination of the rows of $R$, i.e., $\xi = \eta R$ where $\eta \in K^k$. We multiply $\eta$ by its denominator $d \neq 0$ so that $d\eta \in \mathscr{D}^{1\times k}$. Hence $\overline{d\xi} = d\bar{\xi} = 0$ in $M = \mathscr{D}^{1\times l}/\mathscr{D}^{1\times k}R$ and therefore $\bar{\xi} \in \mathrm{T}(M)$.

$\supseteq$: Let $\bar{\xi} \in \mathrm{T}(M)$ therefore there exists $d \neq 0$ with $0 = \text{can}(d\bar{\xi}) = d\text{can}(\bar{\xi})$. Hence $\text{can}(\bar{\xi}) = 0$ since $K \otimes_{\mathscr{D}} M$ is a $K$-vector space.

□

**Definition 6.** Let $\mathcal{B} = \left\{y \in \mathscr{F}^l; R \circ y = 0\right\}$ a behavior. We define the *controllable subbehavior* by

$$\mathcal{B}_{\text{cont}} := (\mathscr{D}^{1\times l} \cap K^{1\times k}R)^{\perp} \subseteq \mathscr{F}^l$$

**Remark 2.** Let $\mathscr{F}$ be an injective cogenerator. Then $\mathcal{B}_{\text{cont}}^{\perp} = \mathscr{D}^{1\times l} \cap K^{1\times k}R$ due to Theorem 3. The torsion submodule of $M$ is the factor module of $\mathcal{B}_{\text{cont}}^{\perp}$ and $\mathcal{B}^{\perp}$, i.e.,

$$\mathrm{T}(M) = (\mathscr{D}^{1\times l} \cap K^{1\times k}R)/\mathscr{D}^{1\times k}R = \mathcal{B}_{\text{cont}}^{\perp}/\mathcal{B}^{\perp}.$$

The orthogonal submodule of $\mathcal{B}$ is a submodule of the orthogonal submodule of $\mathcal{B}_{\text{cont}}$, i.e.,

$$\mathcal{B}^{\perp} \subseteq \mathcal{B}_{\text{cont}}^{\perp}$$

since $\mathcal{B}_{\text{cont}} \subseteq \mathcal{B}$.

**Theorem 4.** *1.* $\mathcal{B}_{\text{cont}}$ *is the biggest controllable subbehavior of* $\mathcal{B}$*.*

*2. For the associated module* $M(\mathcal{B}_{\text{cont}})$ *the following holds:*

$$M(\mathcal{B}_{\text{cont}}) \cong M/\mathrm{T}(M),$$

*i.e.,*

$$M(\mathcal{B}_{\text{cont}}) = \mathscr{D}^{1\times l}/(\mathscr{D}^{1\times l} \cap K^{1\times k}R).$$

*Proof.* 1. Let $\mathcal{B}' \subseteq \mathcal{B}$ be a controllable subbehavior. We show that $\mathcal{B}' \subseteq \mathcal{B}_{\text{cont}}$, i.e., $\mathcal{B}_{\text{cont}}^{\perp} \subseteq \mathcal{B}'^{\perp}$. $\mathcal{B}^{\perp} \subseteq \mathcal{B}'^{\perp}$ since $\mathcal{B}' \subseteq \mathcal{B}$. We consider the canonical map

$$\begin{array}{ccc} \text{can} : M = M(\mathcal{B}) = \mathscr{D}^{1\times l}/\mathcal{B}^{\perp} & \xrightarrow{\text{can}} & M(\mathcal{B}') = \mathscr{D}^{1\times l}/\mathcal{B}'^{\perp} \\ \uparrow & & \uparrow \\ \mathrm{T}(M) = \mathcal{B}_{\text{cont}}^{\perp}/\mathcal{B}^{\perp} & \longrightarrow & \mathrm{T}(M(\mathcal{B}')) = 0 \end{array}$$

And therefore $\mathcal{B}_{\text{cont}}^{\perp} \subseteq \mathcal{B}'^{\perp}$.

2. $M/\mathrm{T}(M) = \left(\mathscr{D}^{1\times l}/\mathcal{B}^{\perp}\right)/\left(\mathcal{B}_{\text{cont}}^{\perp}/\mathcal{B}^{\perp}\right) = \mathscr{D}^{1\times l}/\mathcal{B}_{\text{cont}}^{\perp} = M(\mathcal{B}_{\text{cont}})$. $\mathcal{B}_{\text{cont}}$ is controllable since $M/\mathrm{T}(M)$ is torsion free.

□

**Definition 7.** Let $U \subseteq K^{1\times l}$ be K-subspace of $K^{1\times l}$. We define its *orthogonal subspace* by

$$U^{\perp} = \left\{\tilde{w} \in K^l; \forall \tilde{\xi} \in U : \tilde{\xi}\tilde{w} = 0\right\}.$$

We define the *transfer space of* $\mathcal{B}$ by

$$\tilde{\mathcal{B}} := (K\mathcal{B}^{\perp})^{\perp} = \left\{\tilde{w} \in K^l; R\tilde{w} = 0\right\} \subseteq K^l.$$

**Lemma 4.** *Let* $M = \mathscr{D}^{1\times l}/\mathscr{D}^{1\times k}R$*.*

$$\begin{array}{ccccccc} \mathrm{Hom}_{\mathscr{D}}(M,K) & \cong & \mathrm{Hom}_K(K\otimes M, K) & \cong & \mathrm{Hom}_K(K^{1\times l}/K^{1\times k}R, K) & \cong & \tilde{\mathcal{B}} \\ \varphi_1 & \leftrightarrow & \varphi_2 & \leftrightarrow & \varphi_3 & \leftrightarrow & \tilde{w} \end{array}$$

*where*

$$\varphi_1(\bar{\xi}) = \varphi_2(1\otimes\bar{\xi}) = \varphi_3(\bar{\xi}),\ \widetilde{w}_i = \varphi_1(\bar{\delta}_i) = \varphi_3(\bar{\delta}_i).$$

**Lemma 5.** *The quotient field* $K = \mathrm{quot}(\mathscr{D})$ *is an injective* $\mathscr{D}$*-module.*

*Proof.* We have to show that for every $\mathscr{D}$-module $N$ and all its submodules $M$

$$\mathrm{Hom}_{\mathscr{D}}(\mathrm{inj}, K) : \mathrm{Hom}_{\mathscr{D}}(N,K) \to \mathrm{Hom}_{\mathscr{D}}(M,K)$$

is surjective. But

$$\mathrm{Hom}_K(K\otimes\mathrm{inj}, K) : \mathrm{Hom}_K(K\otimes N,K) \to \mathrm{Hom}_K(K\otimes M,K)$$

is surjective and

$$\mathrm{Hom}_{\mathscr{D}}(N,K) \cong \mathrm{Hom}_K(K\otimes N,K),\ \mathrm{Hom}_{\mathscr{D}}(M,K) \cong \mathrm{Hom}_K(K\otimes M,K).$$

□

**Lemma 6.** *Let* $\mathscr{F}$ *be an injective cogenerator and let* $\mathcal{B}_1, \mathcal{B}_2 \subset \mathscr{F}^l$ *be two behaviors.*

1. *Their intersection is the orthogonal submodule of the sum of their associated submodules, i.e.,*
$$\mathcal{B}_1 \cap \mathcal{B}_2 = (\mathcal{B}_1^\perp + \mathcal{B}_2^\perp)^\perp$$

2. *Their sum is the orthogonal submodule of the intersection of their associated submodules, i.e.,*
$$\mathcal{B}_1 + \mathcal{B}_2 = (\mathcal{B}_1^\perp \cap \mathcal{B}_2^\perp)^\perp$$

*Proof.* 1. By definition.

2. We consider the exact sequence

$$\begin{array}{ccccc} 0 \leftarrow \mathscr{D}^{1\times l}/(\mathcal{B}_1^\perp + \mathcal{B}_2^\perp) & \xleftarrow{\binom{I_l}{-I_l}} & \mathscr{D}^{1\times l}/\mathcal{B}_1^\perp \times \mathscr{D}^{1\times l}/\mathcal{B}_2^\perp & \xleftarrow{(I_l, I_l)_{\mathrm{ind}}} & \mathscr{D}^{1\times l}/(\mathcal{B}_1^\perp \cap \mathcal{B}_2^\perp) \leftarrow 0 \\ & & \| & & \\ & & \mathscr{D}^{1\times l}/(\mathcal{B}_1^\perp \times \mathcal{B}_2^\perp) & & \\ & & (\bar{\xi}, \bar{\xi}) & \longleftarrow\!\shortmid & \bar{\xi} \\ \overline{\xi - \eta} & \longleftarrow\!\shortmid & (\bar{\xi}, \bar{\eta}) & & \end{array}$$

Since $\mathscr{F}$ is an injective cogenerator the duality functor is exact and hence the dual sequence is exact too. For every submodule $U \subset \mathscr{D}^{1\times l}$ of $\mathscr{D}^{1\times l}$ the dual module of the quotient $\mathscr{D}^{1\times l}/U$ is isomorphic to $U^\perp$, i.e.,

$$D(\mathscr{D}^{1\times l}/U) = \mathrm{Hom}(\mathscr{D}^{1\times l}/U, \mathscr{F}) \cong U^\perp.$$

Hence the sequence

$$\begin{array}{ccccccc} 0 \longrightarrow & (\mathcal{B}_1^\perp + \mathcal{B}_2^\perp)^\perp & \xrightarrow{\binom{I_l}{-I_l}} & \mathcal{B}_1 \times \mathcal{B}_2 & \xrightarrow{(I_l, I_l)} & (\mathcal{B}_1^\perp \cap \mathcal{B}_2^\perp)^\perp & \longrightarrow 0 \\ & \| & & & & & \\ & \mathcal{B}_1 \cap \mathcal{B}_2 & & & & & \end{array}$$

is exact too. Hence the map $(I_l, I_l) : \mathcal{B}_1 \times \mathcal{B}_2 \to (\mathcal{B}_1^\perp \cap \mathcal{B}_2^\perp)^\perp$ is surjective. Therefore

$$(\mathcal{B}_1^\perp \cap \mathcal{B}_2^\perp)^\perp = \mathrm{im}((I_l, I_l)) = \mathcal{B}_1 + \mathcal{B}_2$$

as claimed.

□

**Lemma 7.** *Let*

$$\mathcal{B}_1 \xrightarrow{P\circ} \mathcal{B}_2 \xrightarrow{Q\circ} \mathcal{B}_3$$

*be an exact sequence of behaviors. The sequence*

$$\tilde{\mathcal{B}}_1 \xrightarrow{P\circ} \tilde{\mathcal{B}}_2 \xrightarrow{Q\circ} \tilde{\mathcal{B}}_3$$

*is exact too.*

*Proof.* The duality functor is exact since $\mathscr{F}$ is injective. Hence the sequence

$$M(\mathcal{B}_1) \xleftarrow{(\circ P)_{\mathrm{ind}}} M(\mathcal{B}_2) \xleftarrow{(\circ Q)_{\mathrm{ind}}} M(\mathcal{B}_3)$$

is exact. The functor $\mathrm{Hom}_{\mathscr{D}}(-, K)$ is exact since ${}_{\mathscr{D}}K$ is injective. Hence the sequences

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathscr{D}}(M(\mathcal{B}_1), K) & \xrightarrow{\mathrm{Hom}_{\mathscr{D}}(\circ P, K)} & \mathrm{Hom}_{\mathscr{D}}(M(\mathcal{B}_2), K) & \xrightarrow{\mathrm{Hom}_{\mathscr{D}}(\circ Q, K)} & \mathrm{Hom}_{\mathscr{D}}(M(\mathcal{B}_3), K) \\
\updownarrow \cong & & \updownarrow \cong & & \updownarrow \cong \\
\tilde{\mathcal{B}}_1 & \xrightarrow{P\circ} & \tilde{\mathcal{B}}_2 & \xrightarrow{Q\circ} & \tilde{\mathcal{B}}_3
\end{array}$$

are exact too. □

**Lemma 8.** *Let ${}_{\mathscr{D}}M$ be a finitely generated $\mathcal{D}$ module. The following properties are equivalent:*

1. *$M$ is a torsion module.*
2. $\mathrm{Hom}_{\mathscr{D}}(M, K) = 0$.

*Proof.* $M$ torsion $\Leftrightarrow$

$$M = \mathrm{T}(M) \Leftrightarrow K \otimes_{\mathscr{D}} M = 0 \Leftrightarrow 0 = \mathrm{Hom}_K(K \otimes_{\mathscr{D}} M, K) \cong \mathrm{Hom}_{\mathscr{D}}(M, K)$$

□

**Lemma 9.** *The transferspaces of $\mathcal{B}$ and $\mathcal{B}_{\mathrm{cont}}$ are equal, i.e., $\tilde{\mathcal{B}} = \tilde{\mathcal{B}}_{\mathrm{cont}}$*

*Proof.* We consider the exact sequence

$$0 \longrightarrow \mathrm{T}(M) = \mathcal{B}_{\mathrm{cont}}^\perp / \mathcal{B}^\perp \hookrightarrow M = \mathscr{D}^{1\times l}/\mathcal{B}^\perp \xrightarrow{\mathrm{can}} M_{\mathrm{cont}} = \mathscr{D}^{1\times l}/\mathcal{B}_{\mathrm{cont}}^\perp \longrightarrow 0\ .$$

We apply the exact functor $\mathrm{Hom}_{\mathscr{D}}(-, K)$ and get the exact sequence

$$\begin{array}{ccccccc}
0 \longleftarrow & \mathrm{Hom}_{\mathscr{D}}(\mathrm{T}(M), K) & \hookleftarrow & \mathrm{Hom}_{\mathscr{D}}(M, K) & \xleftarrow{\mathrm{Hom}(\mathrm{can}, K)} & \mathrm{Hom}_{\mathscr{D}}(M_{\mathrm{cont}}, K) & \longleftarrow 0 \\
& \| & & \updownarrow \cong & & \updownarrow \cong & \\
& 0 & & \tilde{\mathcal{B}} & \hookleftarrow & \tilde{\mathcal{B}}_{\mathrm{cont}} &
\end{array}\ .$$

Therefore $\mathrm{Hom}_{\mathscr{D}}(M, K) \cong \mathrm{Hom}_{\mathscr{D}}(M_{\mathrm{cont}}, K)$ and $\tilde{\mathcal{B}} = \tilde{\mathcal{B}}_{\mathrm{cont}}$. □

**Theorem 5.** *Let $\mathcal{B}_1, \mathcal{B}_2 \subseteq \mathscr{F}^l$ be two behaviors in $\mathscr{F}^l$. Then*

1. *If $\mathcal{B}_1, \mathcal{B}_2 \subseteq \mathscr{F}^l$ are controllable, their sum $\mathcal{B}_1 + \mathcal{B}_2$ is controllable too.*
2. $\mathcal{B}_{1\text{cont}} + \mathcal{B}_{2\text{cont}} \subseteq (\mathcal{B}_1 + \mathcal{B}_2)_{\text{cont}}$
3. $\mathcal{B}_{1\text{cont}} = \mathcal{B}_{2\text{cont}} \Leftrightarrow \tilde{\mathcal{B}}_1 = \tilde{\mathcal{B}}_2$

*Proof.* See [3] □

**Definition 8.** Let $\mathcal{B}_1, \mathcal{B}_2 \subseteq \mathscr{F}^l$ two behaviors. $\mathcal{B}_1$ is *transfer equivalent* $\mathcal{B}_2$, if and only if their controllable subbehaviors $\mathcal{B}_{1\text{cont}} = \mathcal{B}_{2\text{cont}}$ are equal. ($\Leftrightarrow \tilde{\mathcal{B}}_1 = \tilde{\mathcal{B}}_2$)

## 2.3 IO Systems

Let $\mathcal{B} = \left\{w \in \mathscr{F}^l; R \circ w = 0\right\}$ be a behavior. We define $p = \operatorname{rank}(R)$ and $m := l - p = \operatorname{rank}(M(\mathcal{B}))$ and $\tilde{\mathcal{B}} = \left\{\tilde{w} \in K^l; R \circ \tilde{w} = 0\right\}$ its transfer space.

**Lemma 10.** *For a decomposition $J \uplus J' = \{1, \dots, l\}$ of the set of the numbers $1, \dots, l$ the following properties are equivalent.*

1. *The projection $\operatorname{proj}|_{\tilde{\mathcal{B}}} : \tilde{\mathcal{B}} \xrightarrow{\cong} K^{J'}, \tilde{w} \mapsto (\tilde{w}_i)_{i \in J'}$ is an isomorphism.*
2. (a) *$|J'| = m$ and $|J| = p$.*
   (b) *The columns $(R_{-j})_{j \in J}$ of $R$ are $K$-linear independent.*

*Proof.* See [3]. □

Given the sets $J$ and $J'$ we can permute the components of $\tilde{w}$ and the columns of $R$ so that $J$ and $J'$ are of the form

$$J = \{1, \dots, p\}, \; J' = \{p+1, \dots, p+m = l\}.$$

We define $y := (w_j)_{j \in J}$ and $u := (w_j)_{j \in J'}$, i.e, $w = \binom{y}{u}$. Additionally we define the matrices $P := (R_{-j})_{j \in J}$ and $Q := (-R_{-j})_{j \in J'}$, i.e., $R = (P, -Q)$. Hence

$$\mathcal{B} = \left\{w = \begin{pmatrix} y \\ u \end{pmatrix}; R \circ w = (P, -Q) \circ \begin{pmatrix} y \\ u \end{pmatrix} = P \circ y - Q \circ u = 0 \text{ or } P \circ y = Q \circ u\right\}$$

and the transfer space

$$\tilde{\mathcal{B}} = \left\{\tilde{w} = \begin{pmatrix} \tilde{y} \\ \tilde{u} \end{pmatrix} \in K^{p+m}; P \circ \tilde{y} = Q \circ \tilde{u}\right\}.$$

The columns of $P$ are the first $p$ linear independent columns of $R$ since $\operatorname{rank}(R) = p = \operatorname{rank}(P)$.

**Lemma 11.** *Let $J \uplus J' = \{1, \dots, l\}$ be a decomposition like in Lemma 10.*

1. *There exists uniquely a matrix $H \in K^{p \times m}$ where $PH = Q$.*
2. *The inverse of the projection $\operatorname{proj}|_{\tilde{\mathcal{B}}}$ is $(\operatorname{proj}|_{\tilde{\mathcal{B}}})^{-1} : K^m \to \tilde{\mathcal{B}}, \tilde{u} \mapsto \binom{H\tilde{u}}{\tilde{u}}$.*

*Proof.* See [3]. □

**Definition 9.** A decomposition $J \uplus J' = \{1, \dots, l\}$ that satisfies the properties of Lemma 10 is called an *IO-structure of* $\mathcal{B}$.

**Lemma 12.** *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be IO systems with identical IO structures. Let $H_1$ resp. $H_2$ be their transfer matrices. Then*

$$\mathcal{B}_{1\text{cont}} = \mathcal{B}_{2\text{cont}} \Leftrightarrow H_1 = H_2$$

*Proof.* From Theorem 5 we know that $\mathcal{B}_{1\text{cont}} = \mathcal{B}_{2\text{cont}} \Leftrightarrow \tilde{\mathcal{B}}_1 = \tilde{\mathcal{B}}_2$. From Lemma 11 we know that the transferspaces $\tilde{\mathcal{B}}_i$ are isomorphic to $K^m$, i.e., $K^m \cong \tilde{\mathcal{B}}_i, \tilde{u} \mapsto \binom{H_i\tilde{u}}{\tilde{u}}$.

$\Leftarrow$: We denote $H := H_1 = H_2$. $\tilde{\mathcal{B}}_i = \left\{\binom{H\tilde{u}}{\tilde{u}}; \tilde{u} \in K^m\right\}$. Hence $\tilde{\mathcal{B}}_1 = \tilde{\mathcal{B}}_2$ and therefore $\mathcal{B}_{1\text{cont}} = \mathcal{B}_{2\text{cont}}$

$\Rightarrow$: Let $\tilde{u} \in K^m$ and $y := H_1\tilde{u}$. Hence

$$\begin{pmatrix}\tilde{y}\\ \tilde{u}\end{pmatrix} = \begin{pmatrix}H_1\tilde{u}\\ \tilde{u}\end{pmatrix} \in \tilde{\mathcal{B}}_1 = \tilde{\mathcal{B}}_2$$

Hence there exists a $\tilde{v} \in K^m$ where $\binom{H_1\tilde{u}}{\tilde{u}} = \binom{H_2\tilde{v}}{\tilde{v}}$. Consequently $\tilde{u} = \tilde{v}$ and $H_1\tilde{u} = H_2\tilde{u}$ for all $\tilde{u} \in K^m$. Therefore $H_1 = H_2$.

□

**Theorem 6.** *Let $\mathscr{D}$ be a noetherian integral domain and $K = \mathrm{quot}(\mathscr{D})$ its quotient field. We consider the matrix $H \in K^{p\times m}$. There is a unique controllable IO system $\mathcal{B}$ with transfer matrix $H$.*

*Proof.*

1. First we show that the system $\mathcal{B}$ is unique. Let $\mathcal{B}_1$, $\mathcal{B}_2$ be two controllable IO systems with transfermatrices $H_1 = H_2 = H$. Then $\mathcal{B}_1 = \mathcal{B}_2$ as we have seen in Lemma 12.

2. Now we describe an algorithm to compute $R = (P, -Q)$ from $H$:
Let $0 \neq d \in \mathscr{D}$ where $dH \in \mathscr{D}^{p\times m}$. Hence $d\binom{H}{\mathrm{id}_m} = \binom{dH}{d\mathrm{id}_m} \in \mathscr{D}^{(p+m)\times m}$. Let $R = (P, -Q)$ be universal with $(P, -Q)\binom{dH}{d\mathrm{id}_m} = 0$

$$\Leftrightarrow PdH = Qd\mathrm{id}_m \Leftrightarrow PH = Q \Leftrightarrow (P, -Q)\begin{pmatrix}H\\ \mathrm{id}_m\end{pmatrix} = 0$$

3. We have to show that associated behavior $\mathcal{B}$ to $R = (P, -Q)$ is controllable, i.e., $M(\mathcal{B})$ is torsionfree. The rowmodule of $R$ is the kernel of the map $\circ\binom{dH}{d\mathrm{id}_m}$ : $\mathscr{D}^{p+m} \longrightarrow \mathscr{D}^m \subseteq K^m$ as we have seen above. Consequently

$$\mathscr{D}^{1\times(p+m)}/\mathscr{D}^{1\times k}(P, -Q) \xrightarrow{\binom{H}{\mathrm{id}_m}} K^{1\times m}$$

is a monomorphism by the homomorphism theorem.
Therefore $M = \mathscr{D}^{1\times(p+m)}/\mathscr{D}^{1\times k}(P, -Q)$ is torsionfree.

4. We have to show that the $\mathrm{rank}(P) = p$, i.e. $\{1, \ldots, p\} \uplus \{p+1, \ldots, p+m\}$ is an IO structure of $\mathcal{B}$. Consider the diagram

$$\begin{tikzcd}
\mathscr{D}^{1\times p} \arrow[r, "\circ H"] \arrow[d, "\mathrm{can}"'] \arrow[dr, "\varphi"] & K^{1\times m} \arrow[d, "\mathrm{can}"] \\
\mathscr{D}^{1\times p}/\ker(\varphi) \arrow[r, dashed, "\varphi_{\mathrm{ind}}", "\mathrm{Mono}"'] & K^{1\times m}/\mathscr{D}^{1\times m}
\end{tikzcd}$$

We show that $\ker(\varphi) = \mathscr{D}^{1\times k}P$.

$$\begin{aligned}\overline{\xi H} = 0 \quad &\Leftrightarrow \quad \xi H = \eta \in \mathscr{D}^{1\times m} \Leftrightarrow (\xi, -\eta) \in \ker\left(\begin{pmatrix} H \\ \mathrm{id}_m \end{pmatrix}\right) = \mathscr{D}^{1\times k}(P, -Q) \\ &\Leftrightarrow \quad \exists \zeta : (\xi, -\eta) = \zeta(P, -Q) = \zeta P(\mathrm{id}_p, -H) \Leftrightarrow \xi = \zeta P \\ &\Leftrightarrow \quad \xi \in \mathscr{D}^{1\times k}P\end{aligned}$$

Let $\xi \in \mathscr{D}^{1\times p}/\ker(\varphi)$ then $d\bar{\xi} = \overline{d\xi} = 0$ since $dH \in \mathscr{D}^{p\times m}$. Hence $\mathscr{D}^{1\times p}/\ker(\varphi)$ is a torsion module. Therefore $0 = \operatorname{rank}\left(\mathscr{D}^{1\times p}/\mathrm{Z}(P)\right) = p - \operatorname{rank}(P)$ and consequently $\operatorname{rank}(P) = p$.

□

# 3 The modules $\mathcal{F}_+$ and $\mathcal{G}_+$

In this section we introduce the modules $\mathcal{F}_+$ and $\mathcal{G}_+$. They are the $\mathbb{C}[s]$-linear hull of $\mathcal{C}^\infty Y$ resp. $\mathrm{T}(\mathcal{C}^\infty)Y$. We show that they are $\mathbb{C}(s)$-vector spaces and consider proper rational functions and the images of elements in $\mathcal{C}^\infty Y$ by such functions. We denote by

$$\mathcal{D} = \{\varphi \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}); \mathrm{supp}(\varphi) \text{ is compact}\}$$

the space of test functions, i.e., smooth functions with compact support and by

$$\mathcal{D}' = \{T : \mathcal{D} \to \mathbb{C}; T \text{ is linear and continuous}\}$$

the space of distributions, its algebraic and topological dual space. For the module theoretic treatment of systems and behaviors the following properties of $\mathcal{D}'$ are important.

1. $\mathcal{D}'$ is a $\mathbb{C}[s]$-module with $s \circ y := \frac{dy}{dt}$.
2. $\mathcal{D}'$ is a $\mathcal{C}^\infty = \mathcal{C}^\infty(\mathbb{R}, \mathbb{C})$ module with the multiplication

$$(uT)(\varphi) = T(u\varphi) \text{ for } u \in \mathcal{C}^\infty, T \in \mathcal{D}', \varphi \in \mathcal{D}.$$

**Reminder 4** (Derivative of a distribution)**.** Let $T \in \mathcal{D}'$ be a distribution then its derivative

$$T'(\varphi) = (-1)T(\varphi')$$

is defined via the transposed map of the derivation and therefore

$$T^{(k)}(\varphi) = (-1)^k T(\varphi^{(k)})$$

by induction.

At the beginning we will consider systems that start at $t = 0$, perhaps discontinuously.

**Definition 10.** Let $Y(t) := \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$ be the Heaviside function, we define

$$\mathcal{F}_+ := \mathcal{C}^\infty Y + \mathbb{C}[s] \circ \delta = \mathbb{C}[s] \circ (\mathcal{C}^\infty Y).$$

**Theorem 7.** *$\mathcal{F}_+$ is a $\mathbb{C}[s]$- and $\mathcal{C}^\infty$-submodule of $\mathcal{D}'$ and $\mathcal{F}_+ = \mathcal{C}^\infty Y \oplus \oplus_{i=0}^{\infty} \mathbb{C}\delta^{(i)}$.*

*Proof.* Let $uY \in \mathcal{C}^\infty Y$ then

$$s \circ (uY) = (s \circ u)Y + u(s \circ Y) = u'Y + u\delta = u'Y + u(0)\delta \in \mathcal{C}^\infty Y + \mathbb{C}\delta,$$

hence $s \circ (\mathcal{C}^\infty Y) \subseteq \mathcal{F}_+$ and $s \circ (\mathbb{C}[s] \circ \delta) \subseteq \mathcal{F}_+$ and therefore $s \circ \mathcal{F}_+ \subseteq \mathcal{F}_+$, i.e., $\mathcal{F}_+$ is a $\mathbb{C}[s]$-submodule.

Now we will show that $\mathcal{F}_+$ is a $\mathcal{C}^\infty$-submodule of $\mathcal{D}'$ where, of course, $\mathcal{C}^\infty(\mathcal{C}^\infty Y) \subseteq \mathcal{C}^\infty Y$. Let now $u \in \mathcal{C}^\infty$ then

$$\begin{aligned}
u\frac{\delta^{(m)}}{m!}(\varphi) &= \frac{\delta^{(m)}}{m!}(u\varphi) = \frac{(-1)^m}{m!}\delta\left((u\varphi)^{(m)}\right) \\
&= \frac{(-1)^m}{m!}\sum_{i=0}^{m}\binom{m}{i}u^{(i)}(0)\varphi^{(m-i)}(0) \\
&= (-1)^m\sum_{i=0}^{m}\frac{u^{(i)}(0)}{i!}\frac{\varphi^{(m-i)}(0)}{(m-i)!} \\
&= (-1)^m\sum_{i=0}^{m}\frac{u^{(i)}(0)}{i!}(-1)^{i-m}\frac{\delta^{(m-i)}}{(m-i)!}(\varphi),
\end{aligned}$$

hence

$$u\frac{\delta^{(m)}}{m!} = \sum_{i=0}^{m}(-1)^i\frac{u^{(i)}(0)}{i!}\frac{\delta^{(m-i)}}{(m-i)!} \in \mathbb{C}[s] \circ \delta$$

and $\mathcal{C}^\infty(\mathbb{C}[s] \circ \delta) \subseteq \mathbb{C}[s] \circ \delta \Rightarrow \mathcal{C}^\infty\mathcal{F}_+ \subseteq \mathcal{F}_+$, i.e., $\mathcal{F}_+$ is a $\mathcal{C}^\infty$-submodule.

Next we will show that the sum $\mathcal{F}_+ = \mathcal{C}^\infty Y + \sum_{i=0}^{\infty}\mathbb{C}\delta^{(i)}$ is a direct one. As $\delta$ distribution and its derivative have compact support, i.e., they are elements of $\mathcal{E}' = (\mathcal{C}^\infty)'$, we can apply them on $\mathcal{C}^\infty$-functions. As a preparation we consider

$$\left((-t)^l\right)^{(m)}|_{t=0} = \begin{cases} 0 & l \neq m \\ (-1)^m m! & l = m \end{cases}.$$

Hence

$$\frac{\delta^{(m)}}{m!}\left((-t)^l\right) = \frac{(-1)^m}{m!}\delta(\left((-t)^l\right)^{(m)}) = \frac{(-1)^{2m}m!}{m!}\delta_{lm} = \delta_{lm}.$$

As $\mathbb{C}[s] \circ \delta = \sum_{m=0}^{\infty}\mathbb{C}\delta^{(m)}$ we have to show that the $\delta^{(i)}$ are linearly independent. Let

$$0 = w = \sum_{m=0}^{\infty}a_m\delta^{(m)},$$

i.e., $w(\phi) = 0$ for all test functions $\phi \in \mathcal{E}$. We choose $\phi(x) = (-x)^l$ as above and obtain

$$0 = \sum_{m=0}^{\infty}a_m m!\frac{\delta^{(m)}}{m!}\left((-t)^l\right) = a_l l!$$

hence $a_l = 0$ for all $l$.

Finally we show that the intersection $(\mathcal{C}^\infty Y) \cap (\oplus_{i=0}^{\infty}\mathbb{C}\delta^{(i)})$ contains zero only.

Let $uY \in (\mathcal{C}^\infty Y) \cap (\oplus_{i=0}^{\infty}\mathbb{C}\delta^{(i)})$, with $u \in \mathcal{C}^\infty$. Assume that $uY \neq 0$, i.e., that there exists $t_0 > 0$ such that $u(t_0) \neq 0$.

Without loss of generality we assume that $u(t) > 0$ for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ and $0 \notin [t_0 - \varepsilon, t_0 + \varepsilon]$. We choose $0 \leq \phi \in \mathcal{D}$ with $\mathrm{supp}(\phi) = [t_0 - \varepsilon, t_0 + \varepsilon]$ and obtain $(uY)(\phi) = 0$ since $uY \in (\oplus_{i=0}^{\infty}\mathbb{C}\delta^{(i)})$ and $0 \notin \mathrm{supp}(\phi)$. Hence

$$0 = \int_{\mathbb{R}} u(t)Y(t)\phi(t)\,dt = \int_0^\infty u(t)\phi(t)\,dt = \int_{t_0-\varepsilon}^{t_0+\varepsilon} u(t)\phi(t)\,dt > 0,$$

a contradiction. Therefore $uY = 0$. $\square$

**Theorem 8.** *The Module $\mathcal{F}_+$ is a $\mathbb{C}(s)$-vector space, i.e., divisible and torsionfree.*

*Proof.* 1. First we show, that the map $s\circ : \mathcal{F}_+ \to \mathcal{F}_+$ is an isomorphism.

(a) We proove that the map is injective. Let $y = uY + \sum_{m=0}^{\infty} \alpha_m \delta^{(m)}$. We assume that $y \in \ker(s\circ)$, i.e

$$0 = s \circ y = u'Y + u(0)\delta + \sum_{m=0}^{\infty} \alpha_m \delta^{(m+1)} \in \mathcal{F}_+.$$

From the preceding theorem we infer

$$u'Y = 0,\ u(0) = 0,\ \alpha_m = 0,\ m = 0, 1, 2, \ldots$$

as the sums are direct. Hence $u'(t) = 0$ for $t \geq 0$ and therefore

$$u(t) = u(0) + \int_0^t u'(\tau)\,d\tau = 0 \text{ for } t \geq 0 \text{ and } uY = 0.$$

Finally we see, that

$$y = \underbrace{uY}_{=0} + \underbrace{\sum_{m=0}^{\infty} \alpha_m \delta^{(m)}}_{=0} = 0.$$

(b) To show that the map is surjective, we construct the inverse image for every element of $\mathcal{F}_+$. Let $z = vY + \sum_{m=0}^{\infty} \beta_m \delta^{(m)}$. We define

$$u(t) := \beta_0 + \int_0^t v(\tau)\,d\tau \text{ for } t \in \mathbb{R} \text{ and } y := uY + \sum_{m=0}^{\infty} \beta_{m+1}\delta^{(m)}.$$

$u \in \mathcal{C}^\infty$ as $v \in \mathcal{C}^\infty$, $u'Y = vY$. Application of $s\circ$ yields

$$s \circ y = u'Y + u(0)\delta + \sum_{m=0}^{\infty} \beta_{m+1}\delta^{(m+1)} = vY + \beta_0\delta + \sum_{m=1}^{\infty} \beta_m \delta^{(m)} = z.$$

2. For arbitrary $\lambda \in \mathbb{C}$ we show that $(s-\lambda)\circ : \mathcal{F}_+ \xrightarrow{\cong} \mathcal{F}_+$ is an isomorphism. For $z \in \mathcal{D}'$ we define $y := ze^{\lambda t} \in \mathcal{D}'$. If we apply $s$ we get

$$s \circ y = (s \circ z)e^{\lambda t} + \lambda z e^{\lambda t} = (s \circ z)e^{\lambda t} + \lambda y$$

and therefore

$$(s-\lambda) \circ (ze^{\lambda t}) = (s \circ z)e^{\lambda t}.$$

Therefore the diagram

$$\begin{tikzcd} \mathcal{D}' \arrow[r, "(s-\lambda)\circ"] & \mathcal{D}' \\ \mathcal{D}' \arrow[u, "\cdot e^{\lambda t}", "\cong"'] \arrow[r, "s\circ"] & \mathcal{D}' \arrow[u, "\cdot e^{\lambda t}", "\cong"'] \end{tikzcd}$$

is commutative. This implies that

$$\begin{tikzcd} \mathcal{F}_+ \arrow[r, "(s-\lambda)\circ"] & \mathcal{F}_+ \\ \mathcal{F}_+ \arrow[u, "\cdot e^{\lambda t}", "\cong"'] \arrow[r, "s\circ", "\cong"'] & \mathcal{F}_+ \arrow[u, "\cdot e^{\lambda t}", "\cong"'] \end{tikzcd}$$

is commutative as well and this proves $(s-\lambda)\circ : \mathcal{F}_+ \cong \mathcal{F}_+$ due 1b.

3. For a nonzero complex polynomial $f \in \mathbb{C}[s]$ the fundamental theorem of algebra states that f is of the form

$$f = a \prod_{\lambda \in V_{\mathbb{C}}(f)} (s-\lambda)^{\mu(\lambda)} = a \prod_{i=1}^{n=\deg(f)} (s-\lambda_i)^{\mu_i}, 0 \neq a \in \mathbb{C}.$$

Hence for $v \in \mathcal{F}_+$ holds

$$\begin{aligned} f \circ v &= a((s-\lambda_n)^{\mu_n} \cdots (s-\lambda_1)^{\mu_1}) \circ v \\ &= a((s-\lambda_n)^{\mu_n} \cdots (s-\lambda_2)^{\mu_2}(s-\lambda_1)^{\mu_1-1}) \circ \underbrace{((s-\lambda_1)\circ v)}_{\in \mathcal{F}_+}, \end{aligned}$$

since the scalar multiplication is associative. Therefore $f\circ : V \to V$ is the composition of isomorphisms and hence an isomorphism itself.

□

**Corollary 1.** *Let $h(s) = \frac{q(s)}{p(s)} \in \mathbb{C}(s)$ be a rational function in s, where $p, q \in \mathbb{C}[s]$, $p \neq 0$ are polynomials. Then for arbitrary $u \in \mathcal{F}_+$ the distribution $y := h \circ u$ is the unique solution in $\mathcal{F}_+$ of $p \circ y = q \circ u$.*

*Proof.* $p\circ : \mathcal{F}_+ \xrightarrow{\cong} \mathcal{F}_+$ and hence there is a unique $y \in \mathcal{F}_+$ with $p\circ y = q \circ u$. Since $\mathcal{F}_+$ is a $\mathbb{C}(s)$-vector space, we conclude $y = p^{-1}\circ p \circ y = p^{-1} \circ q \circ u = (p^{-1}q)\circ u = h \circ u$. □

In the sequel we will use the notation

$$\mathcal{F}_{+,1} := \mathcal{C}^\infty Y \text{ and } \mathcal{F}_0 := \mathbb{C}[s]\circ\delta, \text{ hence } \mathcal{F}_+ = \mathcal{F}_{+,1} \oplus \mathcal{F}_0$$

The sequence

$$0 \longrightarrow \mathcal{F}_{+,1} \xrightarrow{inj} \mathcal{F}_+ \xrightarrow{proj} \mathcal{F}_0 \longrightarrow 0$$

is exact.

**Corollary 2.** *Let $0 \neq f \in \mathbb{C}[s]$ a nonzero polynomial and $y \in \mathcal{C}^\infty Y$. Then the image of $y$ by the inverse map $f^{-1}$ is an element of $\mathcal{C}^\infty Y$, i.e., $f^{-1} \circ y \in \mathcal{C}^\infty Y$.*

*Proof.* Let $y = vY \in \mathcal{C}^\infty Y$ and $\lambda \in \mathbb{C}$ be arbitrary then there exists an unique $u_1 \in \mathcal{F}_+$ with $(s-\lambda)\circ u_1 = y$ since $\mathcal{F}_+$ is a $\mathbb{C}(s)$-vector space. We define

$$\tilde{u}(t) := (e^{\lambda t} \int_0^t v(\tau) e^{-\lambda\tau}\, d\tau) Y(t) \in \mathcal{C}^\infty Y,$$

hence

$$\begin{aligned} (s\circ\tilde{u})(t) &= \lambda e^{\lambda t}\left(\int_0^t v(\tau)e^{-\lambda\tau}\,d\tau\right) Y(t) + e^{\lambda t} v(t) e^{-\lambda t} Y(t) \\ &+ \underbrace{\left(e^{\lambda t}\int_0^t v(\tau)e^{-\lambda\tau}\,d\tau\right)\delta}_{=\left(e^{\lambda 0}\int_0^0 v(\tau)e^{-\lambda\tau}\,d\tau\right)=0} \\ &= \lambda\tilde{u}(t) + v(t)Y(t) = \lambda\tilde{u}(t) + y(t), \end{aligned}$$

consequently $(s-\lambda)\circ\tilde{u} = y$ and therefore $u_1 = \tilde{u} \in \mathcal{C}^\infty Y$. By induction over all factors of $f$ there exists $u \in \mathcal{C}^\infty Y$ where $f \circ u = y$ and as $\mathcal{F}_+$ is a $\mathbb{C}(s)$-vector space we finally get

$$u = f^{-1}\circ y \in \mathcal{C}^\infty Y.$$

□

**Reminder 5.** We consider $\mathcal{C}^\infty$ with the $\mathbb{C}[s]$-module structure. The set

$$\mathcal{P} = \{s-\lambda; \lambda \in \mathbb{C}\}$$

is a representative system of primes of the univariate polynomials $\mathbb{C}[s]$. The corresponding primary components of $T(\mathcal{C}^\infty)$ are

$$T_{(s-\lambda)}(\mathcal{C}^\infty) = \{v \in \mathcal{C}^\infty; \exists k \in \mathbb{N} : (s-\lambda)^k \circ v = 0\} = \cup_{k=0}^\infty \mathbb{C}[t]_{<k} e^{\lambda t} = \mathbb{C}[t]e^{\lambda t}.$$

The torsion submodule of $\mathcal{C}^\infty$ has the following primary decomposition

$$T(\mathcal{C}^\infty) = \oplus_{p\in\mathcal{P}} T_p(\mathcal{C}^\infty) = \oplus_{\lambda\in\mathbb{C}} \mathbb{C}[t]e^{\lambda t},$$

i.e., the torsion submodule $T(\mathcal{C}^\infty) = \oplus_{\lambda\in\mathbb{C}}\mathbb{C}[t]e^{\lambda t}$ consists of the polynomial exponential functions.

**Definition and Theorem 9.** *The module $\mathcal{G}_+ := T(\mathcal{C}^\infty)Y \oplus \mathbb{C}[s]\circ\delta$ is a $\mathbb{C}(s)$-subspace of $\mathcal{F}_+$, i.e., it is divisible and torsionfree.*

*Proof.* 1. It is easy to check that $\mathcal{G}_+$ is a $T(\mathcal{C}^\infty)$- and a $\mathbb{C}[s]$-submodule of $\mathcal{F}_+$.

2. We show that the map $s\circ : \mathcal{G}_+ \to \mathcal{G}_+$ is an isomorphism:

   (a) The map $s\circ : \mathcal{G}_+ \to \mathcal{G}_+$ is injective, since $s\circ : \mathcal{F}_+ \to \mathcal{F}_+$ is injective.

   (b) To show that the map is surjective, we construct the inverse image of every element in $T(\mathcal{C}^\infty)$:
   The map
   $$s\circ : T(\mathcal{C}^\infty) \to T(\mathcal{C}^\infty), f \mapsto s\circ f = f'$$
   is surjective since $\mathrm{T}(\mathcal{C}^\infty)$ is divisible. If $u = gY \in T(\mathcal{C}^\infty)Y$ there exists a function $f \in T(\mathcal{C}^\infty)$ with its derivate $s\circ f = f' = g$. We define
   $$y := (f - f(0))\,Y \in T(\mathcal{C}^\infty)Y,$$
   hence
   $$s\circ y = (f' - f(0)')\,Y + (f-f(0))\,\delta = f'Y = gY.$$
   Every element in $\mathcal{G}_+$ is of the form $gY + g_0$ where $g \in \mathrm{T}(\mathcal{C}^\infty)$ and $g_0 \in \mathbb{C}[s]\circ\delta$. Every element
   $$g_0 = \sum_{i=0}^\infty \alpha_i \delta^{(i)} \in \mathbb{C}[s]\circ\delta$$
   has the inverse image
   $$\alpha_0 Y + \sum_{i=0}^\infty \alpha_{i+1}\delta^{(i)} \in \mathcal{G}_+.$$
   Thus $s\circ : \mathcal{G}_+ \to \mathcal{G}_+$ is surjective.

   Therefore $s\circ : \mathcal{G}_+ \to \mathcal{G}_+$ is an isomorphism.

3. Let $f \in \mathbb{C}[s]$ be a polynomial in $s$. The diagram

$$\begin{array}{ccc} \mathcal{G}_+ & \xrightarrow{(s-\lambda)\circ} & \mathcal{G}_+ \\ {\scriptstyle \cdot e^{\lambda t}}\uparrow{\scriptstyle\cong} & & {\scriptstyle \cdot e^{\lambda t}}\uparrow{\scriptstyle\cong} \\ \mathcal{G}_+ & \xrightarrow[\cong]{s\circ} & \mathcal{G}_+ \end{array}$$

   is commutative since $e^{\lambda t}\mathcal{G}_+ \subseteq \mathcal{G}_+$ is a $T(\mathcal{C}^\infty)$-submodule of $\mathcal{G}_+$. Therefore $f\circ : \mathcal{G}_+ \to \mathcal{G}_+$ is an isomorphism, as we have seen in part 3 of the proof of Theorem 8.

□

In the sequel we will use the following notation analogous to $\mathcal{F}_+$:

$$\mathcal{G}_{+,1} = T(\mathcal{C}^\infty)Y,\ \mathcal{G}_0 = \mathcal{F}_0 = \mathbb{C}[s] \circ \delta \text{ and } \mathcal{G}_+ = \mathcal{G}_{+,1} \oplus \mathcal{G}_0.$$

The sequence

$$0 \longrightarrow \mathcal{G}_{+,1} \xrightarrow{inj} \mathcal{G}_+ \xrightarrow{proj} \mathcal{G}_0 \longrightarrow 0$$

is exact too.

**Definition 11.** We define the ring of proper rational functions

$$\mathbb{C}(s)_{pr} = \mathbb{C}(s) \cap \mathbb{C}[[s^{-1}]] = \left\{ r(s) = \frac{q(s)}{p(s)} \in \mathbb{C}(s);\ \deg(r) = \deg(q) - \deg(p) \leq 0 \right\}$$

as the intersection of the rational functions in $s$ and the ring of formal power series in $s^{-1}$. Additionally we define its ideal of strictly proper rational functions

$$\mathbb{C}(s)_{spr} = \mathbb{C}(s) \cap \mathbb{C}[[s^{-1}]]s^{-1} = \{r \in \mathbb{C}(s); deg(r) < 0\}$$

as the intersection of the field of rational functions in $s$ and the ideal of formal power series in $s^{-1}$ with constant coefficient zero.

**Reminder 6.** The field of rational functions can be written in the following form

$$\mathbb{C}(s) = \mathbb{C}[s] \oplus \mathbb{C}(s)_{spr}.$$

By partial fraction decomposition we get the following representation of the strictly proper rational functions

$$\mathbb{C}(s)_{spr} = \oplus_{\lambda \in \mathbb{C}} \oplus_{\mu=1}^{\infty} \mathbb{C} \frac{1}{(s-\lambda)^\mu}$$

and therefore

$$\mathbb{C}(s)_{pr} = \mathbb{C} \oplus \oplus_{\lambda \in \mathbb{C}} \oplus_{\mu=1}^{\infty} \mathbb{C} \frac{1}{(s-\lambda)^\mu} = \mathbb{C} \oplus \mathbb{C}(s)_{spr}$$

**Theorem 10.** *The image of $\mathcal{F}_{+,1}$ by every proper rational function is in $\mathcal{F}_{+,1}$, i.e,*

$$\mathbb{C}(s)_{pr} \circ \mathcal{F}_{+,1} \subseteq \mathcal{F}_{+,1}$$

*Proof.* Let $r \in \mathbb{C}(s)_{spr}$ be a stricly proper rational function. Then it is of the form

$$r = a \sum_{i=1}^{m} \sum_{\mu=1}^{n} c_{i\mu} \frac{1}{(s-\lambda_i)^\mu},\ a \in \mathbb{C}.$$

The image by every factor $(s-\lambda)^{-1} \circ \mathcal{F}_{+,1}$ is in $\mathcal{F}_{+,1}$, as we have seen in Corollary 2. Hence for every $y \in \mathcal{F}_{+,1}$ holds

$$r \circ y = \left( a \sum_{i=1}^{m} \sum_{\mu=1}^{n} c_{i\mu} \frac{1}{(s-\lambda_i)^\mu} \right) \circ y = a \sum_{i=1}^{m} \sum_{\mu=1}^{n} c_{i\mu} \frac{1}{(s-\lambda_i)^{\mu-1}} \circ \underbrace{\left((s-\lambda_i)^{-1} \circ y\right)}_{\in \mathcal{F}_{+,1}}$$

and therefore by induction $r \circ y \in \mathcal{F}_{+,1}$. $\square$

**Example 3.** We consider $y = \cos(t)\, Y \in \mathcal{G}_{+,1} \subset \mathcal{F}_{+,1}$.
$h = \frac{s^2-s}{s^2-4}$ is a proper rational function and therefore

$$h \circ y = \frac{1}{5}\left(\cos(t) - \sin(t) + \left(1 + 3\, e^{-4\,t}\right) e^{2\,t}\right) Y$$

is an element of $\mathcal{G}_{+,1} \subset \mathcal{F}_{+,1}$.
$r = \frac{s^2-s}{s-4}$ is not proper and

$$r \circ y = \left(\frac{48}{17}\, e^{4\,t} + \frac{3}{17}\, \cos(t) - \frac{5}{17}\, \sin(t)\right) Y + \delta$$

is not in $\mathcal{F}_{+,1}$.

**Theorem 11.** *For a rational function $H \in \mathbb{C}(s)$ the following properties are equivalent:*

1. *$H \in \mathbb{C}(s)_{pr}$ is a proper rational function.*
2. *The image of $\mathcal{F}_{+,1}$ by $H$ is a $\mathbb{C}(s)$-subspace of $\mathcal{F}_{+,1}$, i.e. $H \circ \mathcal{F}_{+,1} \subseteq \mathcal{F}_{+,1}$.*

*Proof.*1.⇒2. By the preceding theorem.

2.⇒1. We split

$$H = H_{pol+} + H_{pr}$$

into a polynomial part $H_{pol+} \in \mathbb{C}[s]s$ and a proper part $H_{pr} \in \mathbb{C}(s)_{pr}$. We assume that $H \circ \mathcal{F}_{+,1} \subseteq \mathcal{F}_{+,1}$. Hence $H_{pol+} \circ \mathcal{F}_{+,1} \subseteq \mathcal{F}_{+,1}$, since $H_{pr} \circ \mathcal{F}_{+,1} \subseteq \mathcal{F}_{+,1}$.
We assume that

$$H_{pol+} = a_m s^m + \ldots + a_1 s = s\,\underbrace{\left(a_m s^{m-1} + \ldots a_1\right)}_{=:f} \neq 0$$

is not zero. For all functions $y \in \mathcal{F}_{+,1}$ there is a function $u_1 \in \mathcal{F}_{+,1}$ with $f \circ u_1 = y$. Therefore for $y = Y$ there exists a function $u \in \mathcal{F}_{+,1}$ with $f \circ u = Y$. We conclude

$$H_{pol+} \circ u = s \circ f \circ u = s \circ Y = \delta \notin \mathcal{F}_{+,1}$$

a contradiction. Therefore $H_{pol+} = 0$ and $H = H_{pr} \in \mathbb{C}(s)_{pr}$.

□

**Corollary 3.** 1. *$\mathcal{F}_{+,1} \subseteq \mathcal{F}_{+}$ is a $\mathbb{C}(s)_{pr}$- and a $\mathbb{C}[s^{-1}]$-submodule of $\mathcal{F}_{+,1}$.*

2. *For a rational function $H \in \mathbb{C}(s)$ the following is equivalent*
   (a) *$H \in \mathbb{C}(s)_{pr}$ is a proper rational function.*
   (b) *The projection $proj_{\mathcal{F}_0}(H \circ \mathcal{F}_{+,1}) = 0$ on $\mathcal{F}_0 = \mathbb{C}[s] \circ \delta$ is zero.*
3. *For a matrix $H \in \mathbb{C}(s)^{p\times m}$ of rational functions the following is equivalent*
   (a) *$H \in \mathbb{C}(s)_{pr}^{p\times m}$ is a matrix of proper rational functions.*
   (b) *The image of $\mathcal{F}_{+,1}^m$ by $H$, $H \circ \mathcal{F}_{+,1}^m \subseteq \mathcal{F}_{+}^p$, is a $\mathbb{C}[s^{-1}]$-submodule of $\mathcal{F}_{+,1}^p$.*
   (c) *The projection $proj_{\mathcal{F}_0}(H \circ \mathcal{F}_{+,1}^m) = 0$ on $(\mathcal{F}_0 = \mathbb{C}[s] \circ \delta)^p$ is zero.*

# 4 The module $\Delta = \mathbb{C}^{(\mathbb{N})}$

In this chapter we introduce the module $\Delta$ of finite complex valued sequences.

**Reminder 7.** Let $\mathbb{C}^{\mathbb{N}}$ be the set of all complex valued sequences. For every element $y \in \mathbb{C}^{\mathbb{N}}$ the support is defined by

$$\operatorname{supp}(y) = \{i, a_i = a(i) \neq 0\}.$$

We consider $\mathbb{C}^{\mathbb{N}}$ as a $\mathbb{C}[s^{-1}]$-module where the multiplication with $s^{-1}$ is the left shift, i.e.,

$$s^{-1} \circ y = s^{-1} \circ (y(0)y(1)\ldots) = (y(1)y(2)\ldots)$$

resp.

$$(s^{-1} \circ y)(n) = y(n+1).$$

Additionally we define the set

$$\mathbb{C}^{(\mathbb{N})} = \{y \in \mathbb{C}^{\mathbb{N}}; \operatorname{supp}(y) \text{ is finite}\}$$

of the finite complex valued sequences. $\mathbb{C}^{(\mathbb{N})}$ is a $\mathbb{C}$-subspace of $\mathbb{C}^{\mathbb{N}}$ with the basis $\delta_i = (\delta_{ij})_{j\in\mathbb{N}}$ as the reader can check for himself.
The primary components of $T(\mathbb{C}^{\mathbb{N}})$ are

$$T_{(s^{-1}-\lambda)} = \{y \in \mathbb{C}^{\mathbb{N}}; \exists k \in \mathbb{N} : (s^{-1}-\lambda)^k \circ y = 0\} = \begin{cases} \mathbb{C}^{(\mathbb{N})} & \text{for} \quad \lambda = 0 \\ \mathbb{C}[i](\lambda^i)_{i\in\mathbb{N}} & \text{for} \quad \lambda \neq 0 \end{cases}$$

Therefore the torsion submodule $T(\mathbb{C}^{\mathbb{N}})$ is

$$T(\mathbb{C}^{\mathbb{N}}) = \oplus_{\lambda\in\mathbb{C}} T_{(s^{-1}-\lambda)} = \mathbb{C}^{(\mathbb{N})} \oplus \oplus_{0\neq\lambda\in\mathbb{C}} \mathbb{C}[i](\lambda^i)_{i\in\mathbb{N}}.$$

**Remark 3.** We can consider $\mathbb{C}^{(\mathbb{N})}$ both as a $\mathbb{C}[s^{-1}]$- and as a $\mathbb{C}[s]$-module, where

$$s^{-1} \circ \delta_i = \begin{cases} \delta_{i-1} & \text{for} \quad i > 0 \\ 0 & \text{for} \quad i = 0 \end{cases} \text{ and } s \circ \delta_i = \delta_{i+1}.$$

These two multiplications are not commutative since

$$\begin{aligned} s^{-1} \circ (s \circ \delta_0) &= s^{-1} \circ \delta_1 = \delta_0, \text{ but} \\ s \circ (s^{-1} \circ \delta_0) &= s \circ 0 = 0. \end{aligned}$$

**Definition 12.** We define the module $\Delta$ by

$$\Delta := \mathbb{C}^{(\mathbb{N})}$$

the module of all finite complex valued sequences. We consider $\Delta$ with the $\mathbb{C}[s]$- and the $\mathbb{C}[s^{-1}]$-module structure.

**Remark 4.** $\Delta$ is an injective $\mathbb{C}[s^{-1}]$-module as direct summand of $T(\mathbb{C}^{\mathbb{N}})$.

**Lemma 13.** *Consider the diagram*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}_{+,1} \arrow[r, hook] & \mathcal{F}_+ \arrow[rr, "\Phi"] \arrow[dr, "\text{proj}"'] & & \Delta \arrow[r] \arrow[dl, "\cong"] & 0 \\ & & & \mathcal{F}_0 = \mathbb{C}[s] \circ \delta & & \end{tikzcd}.$$

$\mathcal{F}_0$ *is isomorphic to* $\Delta$*, i.e.* $\mathcal{F}_0 = \mathbb{C}[s] \circ \delta \cong \Delta = \mathbb{C}^{(\mathbb{N})}$*, since the map*

$$\mathbb{C}[s] \circ \delta \overset{\cong}{\to} \Delta = \mathbb{C}^{(\mathbb{N})}, \sum a_i \delta^{(i)} \mapsto \sum a_i \delta_i$$

*is an isomorphism. The following properties hold*

1. *The module* $\mathcal{F}_{+,1}$ *is a* $\mathbb{C}[s^{-1}]$*-submodule of* $\mathcal{F}_+$.
2. *The module* $\mathcal{F}_0 = \mathbb{C}[s] \circ \delta$ *is a* $\mathbb{C}[s]$*-submodule of* $\mathcal{F}_+$.
3. *The map* $\Phi : \mathcal{F}_+ \to \Delta = \mathbb{C}^{(\mathbb{N})}$ *is* $\mathbb{C}[s^{-1}]$*-linear but not* $\mathbb{C}[s]$*-linear.*

*Proof.* 1. See Theorem 10.

2. See Theorem 7.

3. To analyse the linearity of $\Phi$ we consider the following diagram

$$\begin{tikzcd} \mathcal{F}_+ \arrow[r, "\Phi"] \arrow[d, "s^{-1}\circ"'] & \Delta \arrow[d, "s^{-1}\circ"] \\ \mathcal{F}_+ \arrow[r, "\Phi"'] & \Delta \end{tikzcd}.$$

(a) First we show that $\Phi$ is $\mathbb{C}[s^{-1}]$-linear.

i. Let $i = 0$ then

$$\begin{tikzcd} \delta = \delta^{(0)} \arrow[r, mapsto, "\Phi"] \arrow[d, mapsto, "s^{-1}\circ"'] & \delta_0 \arrow[d, mapsto, "s^{-1}\circ"] \\ s^{-1} \circ \delta = Y \arrow[r, mapsto, "\Phi"'] & 0 \end{tikzcd}$$

$$\Phi(s^{-1} \circ \delta) = \Phi(Y) = 0 = s^{-1} \circ \delta_0 = s^{-1} \circ \Phi(\delta),$$

i.e., $\Phi(s^{-1} \circ \delta) = s^{-1} \circ \Phi(\delta)$

ii. Let $i > 0$ then

$$\begin{tikzcd} \delta^{(i)} \arrow[r, mapsto, "\Phi"] \arrow[d, mapsto, "s^{-1}\circ"'] & \delta_i \arrow[d, mapsto, "s^{-1}\circ"] \\ \delta^{(i-1)} \arrow[r, mapsto, "\Phi"'] & \delta_{i-1} \end{tikzcd}$$

$$\Phi(s^{-1} \circ \delta^{(i)}) = \Phi(\delta^{(i-1)}) = \delta_{i-1} = s^{-1} \circ \delta_i = s^{-1} \circ \Phi(\delta^{(i)}),$$

i.e., $\Phi(s^{-1} \circ \delta^{(i)}) = s^{-1} \circ \Phi(\delta^{(i)})$

Hence $\Phi$ is $\mathbb{C}[s^{-1}]$-linear.

(b) Now we show that $\Phi$ is not $\mathbb{C}[s]$-linear.

$$\begin{tikzcd} Y \arrow[r, mapsto, "\Phi"] \arrow[d, mapsto, "s\circ"'] & 0 \arrow[d, mapsto, "s\circ"] \\ s \circ Y = \delta \arrow[r, mapsto, "\Phi"] & \delta_0 \neq 0 \end{tikzcd}$$

$$\Phi(s \circ Y) = \Phi(\delta) = \delta_0 \text{ but } s \circ \Phi(Y) = s \circ 0 = 0.$$

Hence $\Phi(s \circ Y) \neq s \circ \Phi(Y)$ and therefore $\Phi$ is not $\mathbb{C}[s]$-linear.

$\square$

**Corollary 4.** *The sequence*

$$0 \longrightarrow \mathcal{F}_{+,1} \hookrightarrow \mathcal{F}_{+} \xrightarrow{\Phi} \Delta = \mathbb{C}^{(\mathbb{N})} \longrightarrow 0$$

*is an exact sequence of* $\mathbb{C}[s^{-1}]$*-modules as* $\Phi(y) = 0$ *for* $y \in \mathcal{F}_{+,1}$.

**Remark 5.** $\mathcal{F}_+$ is a $\mathbb{C}(s)$-vector space and $\mathcal{F}_{1,+}$ is a $\mathbb{C}(s)_{pr}$-submodule of $\mathcal{F}_+$.

# 5 The Impulsive Behavior of IO-Systems

In this section we introduce the concept of impulsive behavior of input-output systems as a submodule of $\Delta^p$ and the impulsive solutions of an input-output system as the inverse image of the impulsive behavior by the map $\Phi = \text{proj} : \mathcal{F}_+^p \to \Delta^p$. We give an algorithm to compute the impulsive behavior of a given transfer matrix of an input-output system. We discuss the structure of the impulsive behavior and give a formula to compute the impulsive output of a given input.

**Definition and Corollary 1.** Let $H \in \mathbb{C}(s)^{p\times m}$ be a matrix of rational functions. Consider the diagram

$$\begin{tikzcd}
\mathcal{F}_+^m \arrow[r, "H\circ"] & \mathcal{F}_+^p \arrow[r, "\Phi"] & \Delta^p \\
\mathcal{F}_{+,1}^m \arrow[u, hook] \arrow[r, "H\circ"'] & H\circ\mathcal{F}_{+,1}^m \arrow[u, hook] \arrow[r] & \Phi(H\circ\mathcal{F}_{+,1}^m) \arrow[u, hook]
\end{tikzcd}$$

We call $\mathcal{B}_\infty := \Phi(H \circ \mathcal{F}_{+,1}^m)$ the impulsive behavior of $H$. The following properties hold:

1. The image of $H \circ \mathcal{F}_{+,1}^m$ by $\Phi$ is a $\mathbb{C}[s^{-1}]$-submodule of $\Delta$, i.e.,

$$\Phi(H \circ \mathcal{F}_{+,1}^m) \subseteq_{\mathbb{C}[s^{-1}]} \Delta.$$

2. The matrix $H$ is proper if and only if the image of $\mathcal{F}_{+,1}^m$ by $H$ is in $\mathcal{F}_{+,1}$. This is the case if and only if the impulsive behavior of $H$ is zero, i.e.

$$H \in \mathbb{C}(s)_{pr}^{p\times m} \Leftrightarrow H\circ\mathcal{F}_{+,1}^m \subseteq \mathcal{F}_{+,1}^p \Leftrightarrow \mathcal{B}_\infty = \Phi(H\circ\mathcal{F}_{+,1}^m) = 0$$

**Reminder 8.** Let $H \in \mathbb{C}(s)^{p\times m}$ a matrix of rational functions. We define the matrix $P_{\text{cont}} \in \mathbb{C}[s]^{p\times p}$ by

$$\mathbb{C}[s]^{1\times p} P_{\text{cont}} = \left\{\xi \in \mathbb{C}[s]^{1\times p}; \xi H \in \mathbb{C}[s]^{1\times m}\right\}.$$

Additionally we define the matrix $Q_{\text{cont}} := P_{\text{cont}} H \in \mathbb{C}[s]^{p\times m}$. Then the rank of the matrix $(P_{\text{cont}}, -Q_{\text{cont}})$ is $p$, i.e.,

$$\text{rank}(P_{\text{cont}}) = \text{rank}((P_{\text{cont}}, -Q_{\text{cont}})) = p.$$

The module $\mathbb{C}[s]^{1\times p}(P_{\text{cont}}, -Q_{\text{cont}})$ is a direct summand of $\mathbb{C}[s]^{1\times(p+m)}$, i.e., exists

$$\begin{pmatrix} Y \\ U \end{pmatrix} \in \mathbb{C}[s]^{(p+m)\times p} \text{ with } (P_{\text{cont}}, -Q_{\text{cont}}) \begin{pmatrix} Y \\ U \end{pmatrix} = P_{\text{cont}}Y - Q_{\text{cont}}U = \text{id}_p.$$

As $\mathbb{C}(s) = \mathbb{C}(s^{-1})$ is the quotient field of both $\mathbb{C}[s]$ and $\mathbb{C}[s^{-1}]$, we can likewise consider $H \in \mathbb{C}(s)^{p\times m}$ as a matrix $H \in \mathbb{C}(s^{-1})$ of rational functions in $s^{-1}$.

**Corollary 5.** *Let $A \in \mathbb{C}[s^{-1}]^{p\times p}$ be defined by*

$$\mathbb{C}[s^{-1}]^{1\times p}A = \left\{\xi \in \mathbb{C}[s^{-1}]^{1\times p}; \xi H \in \mathbb{C}[s^{-1}]^{1\times m}\right\}.$$

*We define*

$$B := AH \in \mathbb{C}[s^{-1}]^{p\times m}$$

*By construction $A \in \mathbb{C}[s^{-1}]^{p\times p}$ is a matrix of polynomials in $s^{-1}$ and $\det(A) \neq 0$. Then the following holds*

$$H = A^{-1}B \text{ and there is } \begin{pmatrix} Y \\ U \end{pmatrix} \in \mathbb{C}[s^{-1}]^{(p+m)\times p}, \text{ where } AY - BU = id_p.$$

**Corollary 6.** *Let $M$ be a $\mathbb{C}[s^{-1}]$-module. We consider the maps $\binom{Y}{U}\circ : M^p \to M^{p+m}$ and $(A,-B)\circ : M^{p+m} \to M^p$. The following properties hold:*

1. *For all $\mathbb{C}[s^{-1}]$-modules the map $\binom{Y}{U}\circ : M^p \to M^{p+m}$ is a right inverse of $(A,-B)\circ M^{p+m} \to M^p$, i.e.,*

$$\begin{array}{ccc} M^{p+m} & \underset{\binom{Y}{U}\circ}{\overset{(A,-B)\circ}{\rightleftarrows}} & M^p \\ w = \begin{pmatrix} y \\ u \end{pmatrix} & \longmapsto & A\circ y - B\circ u \\ \begin{pmatrix} Y\circ v \\ U\circ v \end{pmatrix} & \longleftarrow\!\shortmid & v \end{array}$$

   *with $(A,-B)\circ\binom{Y}{U} = \mathrm{id}_{M^p}$.*

2. *Let $\Phi : M \to N$ be a $\mathbb{C}[s^{-1}]$-linear map. The following diagram is commutative*

$$\begin{tikzcd} M^{p+m} \arrow[r, "\Phi^{p+m}"] \arrow[d, shift left, "(A,-B)\circ"] & N^{p+m} \arrow[d, shift left, "(A,-B)\circ"] \\ M^p \arrow[r, "\Phi^p"] \arrow[u, shift left, "\binom{Y}{U}\circ"] & N^p \arrow[u, shift left, "\binom{Y}{U}\circ"] \end{tikzcd}$$

   *In particular this holds for the map $\Phi$ from Lemma 13.*

We consider the IO system

$$\mathcal{B} = \left\{\begin{pmatrix} y \\ u \end{pmatrix} \in \mathcal{F}^{p+m}; P\circ y = Q\circ u\right\},$$

where $PH = Q$. Let the matrices $A$, $B$ be defined as in Corollary 5. The equations

$$P\circ y = Q\circ u \Leftrightarrow P\circ(y - H\circ u) = 0 \Leftrightarrow y = H\circ u$$

are equivalent to

$$A\circ y = B\circ u \Leftrightarrow y = A^{-1}B\circ u \Leftrightarrow y = H\circ u$$

since $\mathcal{F}_+$ is a $\mathbb{C}(s)$-vector space and $\mathrm{rank}(P) = \mathrm{rank}(A) = p$.

**Remark 6.** The matrices $P, Q$ have coefficients in $\mathbb{C}[s]$ and the matrices $A, B$ have coefficients in $\mathbb{C}[s^{-1}]$. The map $\Phi$,

$$\begin{tikzcd} \Phi : \mathcal{F}_+ \arrow[rr] \arrow[dr, "\mathrm{proj}"'] & & \Delta \\ & \mathbb{C}[s] \circ \delta \arrow[ur, "\cong"'] & \end{tikzcd}$$

$$y_1 = \sum \alpha_i \delta^{(i)} \mapsto \sum \alpha_i \delta_i,$$

is $\mathbb{C}[s^{-1}]$-linear but not $\mathbb{C}[s]$-linear as we have seen in Lemma 13. Therefore the equations $A \circ y = B \circ u$ are advantageous.

**Theorem 12.** *Let $\mathcal{B}$ be an IO system with transfer matrix $H$. Consider the diagram*

$$\begin{tikzcd} \mathcal{F}_+^m \arrow[r, "H\circ"] & \mathcal{F}_+^p = \mathcal{F}_{+,1}^p \oplus \mathcal{F}_0^p \arrow[r, "\Phi"] & \Delta^p \\ \mathcal{F}_{+,1}^m \arrow[u, hook] & & \\ u \arrow[r, mapsto] & H \circ u = (H \circ u)_1 + (H \circ u)_0 \arrow[r, mapsto] & \Phi((H \circ u)_0) \end{tikzcd}$$

*The impulsive behavior of $\mathcal{B}$ resp. $H$ is the kernel of the linear map $A\circ$ on $\Delta^p$, i.e.,*

$$\mathcal{B}_\infty = \ker(A\circ : \Delta^p \to \Delta^p) = \{\tilde{y_0} \in \Delta^p; A \circ \tilde{y_0} = 0\}.$$

*Proof.* $\subseteq$: Let $y = H \circ u = y_1 + y_0$ where $u \in \mathcal{F}_{+,1}^m$, $y_1 \in \mathcal{F}_{+,1}^p$ and $y_0 \in \mathcal{F}_0^p$. Hence $A \circ y = B \circ u$ by construction of $A$ and $B$. The images of $y_1$ and $u$ have no impulsive component, i.e.,

$$A \circ y_1, B \circ u \in \mathcal{F}_{+,1}^p,$$

since $A$ and $B$ are matrices with coefficients in $\mathbb{C}[s^{-1}]$. Consequently

$$\Phi(A \circ y_0) = \Phi(B \circ u - A \circ y_1) = 0.$$

Since $\Phi$ is $\mathbb{C}[s^{-1}]$-linear

$$A \circ \Phi(H \circ u) = A \circ \Phi(y_0) = \Phi(A \circ y_0) = 0$$

as claimed.

$\supseteq$: Let $\tilde{y_0}$ be a nonzero element of $\ker(A\circ)$. It is of the form $\tilde{y_0} = \sum \alpha_i \delta_i$ where $\alpha_i \in \mathbb{C}^p$. We define

$$y_0 := \sum \alpha_i \delta^{(i)} \in \mathcal{F}_0^p \subseteq \mathcal{F}_+^p$$

and obtain $\Phi(y_0) = \tilde{y_0} \neq 0$ but $A \circ \Phi(y_0) = 0$. The diagram

$$\begin{tikzcd} \mathcal{F}_+^p \arrow[r, "\Phi"] \arrow[d, "A\circ"'] & \Delta^p \arrow[d, "A"] \\ \mathcal{F}_+^p \arrow[r, "\Phi"'] & \Delta^p \end{tikzcd}$$

is commutative since $\Phi$ is $\mathbb{C}[s^{-1}]$-linear. Hence $\Phi(A \circ y_0) = A \circ \Phi(y_0) = 0$ and therefore $A \circ y_0$ is not an impulsive solution, i.e., $A \circ y_0 \in \mathcal{F}^p_{+,1}$. We consider the $\mathbb{C}[s^{-1}]$-linear map $(A, -B)\circ$ and its right inverse $\binom{Y}{U}$, i.e.,

$$\mathcal{F}^{p+m}_{+,1} \underset{\binom{Y}{U}\circ}{\overset{(A,-B)\circ}{\rightleftarrows}} \mathcal{F}^p_{+,1}$$

where $(A, -B)\binom{Y}{U} = \mathrm{id}_p$. Let $\binom{y_1}{u_1} := \binom{Y}{U} \circ (A \circ y_0) \in \mathcal{F}^{p+m}_{+,1}$ be the image of $A \circ y_0$ by the right inverse of $(A, -B)$. Hence

$$A \circ y_1 - B \circ u_1 = (A, -B) \circ \begin{pmatrix} y_1 \\ u_1 \end{pmatrix} = (A, -B) \circ \begin{pmatrix} Y \\ U \end{pmatrix} \circ A \circ y_0 = A \circ y_0.$$

Consequently

$$\begin{aligned} A \circ y_1 - B \circ u_1 = A \circ y_0 \quad &\Leftrightarrow \quad A \circ (y_0 - y_1) = B \circ (-u_1) \\ &\Leftrightarrow \quad y_0 - y_1 = H \circ (-u_1). \end{aligned}$$

Hence

$$\tilde{y}_0 = \Phi(y_0 - y_1) = \Phi(H \circ (-u_1)) \in \Phi\left(H \circ \mathcal{F}^m_{+,1}\right),$$

i.e., $\tilde{y}_0$ is in the impulsive behavior of $H$. The element $y_1$ is in the kernel of $\Phi$ since $y_1 \in \mathcal{F}^m_{+,1}$. Consequently $y_0$ is an impulsive solution of $H$ resp. $\mathcal{B}$.

Therefore the impulsive behavior of $H$ is the kernel of the linear map $A\circ$, i.e,

$$\ker(A\circ : \Delta^p \to \Delta^p) = \Phi(H \circ \mathcal{F}^m_{+,1}) = \mathcal{B}_\infty.$$

□

**Lemma 14.** *Let $0 \neq \lambda \in \mathbb{C}$ be a nonzero complex number. Then the annihilator* $\mathrm{ann}_\Delta(s^{-1} - \lambda) = 0$ *consists of zero only.*

*Proof.* Let $y \in \Delta$ be a finite sequence, i.e. there exists a $T \in \mathbb{N}$ such that $y(T) \neq 0$ and $y(t) = 0$ for all $t > T$. We consider the equation $(s^{-1} - \lambda) \circ y = 0$ where $0 \neq \lambda \in \mathbb{C}$ is a nonzero complex number. Hence

$$0 = (s^{-1} \circ y)(T) - \lambda y(T) = \underbrace{y(T+1)}_{=0} + \lambda y(T) = \underbrace{\lambda}_{\neq 0} y(T).$$

Therefore $y(T) = 0$, a contradiction. □

**Lemma 15.** *Let $0 \neq f = \prod_\lambda (s^{-1} - \lambda)^{\mu(\lambda)} \in \mathbb{C}[s^{-1}]$ be a nonzero polynomial. Then its annihilator*

$$\mathrm{ann}_\Delta(f) = \mathrm{ann}_\Delta(s^{-\mu(0)}) = \bigoplus_{i=0}^{\mu(0)-1} \mathbb{C}\delta_i$$

*equals the annihilator of $s^{-\mu(0)}$.*

*Proof.* According to the Chinese Remainder Theorem the annihilator of $f$ has the following direct sum decomposition

$$\begin{aligned} \mathrm{ann}_\Delta(f) &= \oplus_\lambda \mathrm{ann}_\Delta\left((s^{-1} - \lambda)^{\mu(\lambda)}\right) \\ &= \mathrm{ann}_\Delta\left(s^{-\mu(0)}\right) \oplus \oplus_{\lambda \neq 0} \underbrace{\mathrm{ann}_\Delta\left(s^{-1} - \lambda)^{\mu(\lambda)}\right)}_{=0} \\ &= \mathrm{ann}_\Delta\left(s^{-\mu(0)}\right). \end{aligned}$$

□

**Reminder 9** (Smith normal form)**.** Let $\mathscr{D}$ be euclidean, $\mathscr{F}$ divisible and $A \in \mathscr{D}^{m\times n}$. Then there exist two invertible matrices $U \in \mathrm{Gl}_m(\mathscr{D})$, $V \in \mathrm{Gl}_n(\mathscr{D})$ such that $D = UAV = \mathrm{diag}(e_1, \dots, e_r, 0, \dots, 0)$ is a diagonal matrix. We consider the system of equations $A \circ y = 0$ where $y \in \mathscr{F}$. Since $U$ and $V$ are invertible the following equations are equivalent.

$$A \circ y = 0 \Leftrightarrow U \circ A \circ y = 0 \Leftrightarrow U \circ A \circ V \circ V^{-1} \circ y = 0 \Leftarrow D \circ (V^{-1} \circ y) = 0$$

hence the kernel of $A$ is isomorphic to the kernel of the diagonal matrix $D$, i.e.,

$$\begin{array}{rcl} \ker(A) = \{y; A \circ y = 0\} & \overset{\cong}{\longleftrightarrow} & \{\tilde{y}; D \circ \tilde{y} = 0\} = \ker(D). \\ y & \longmapsto & V^{-1} \circ y \\ V \circ \tilde{y} & \longleftarrow & \tilde{y} \end{array}$$

Since $D$ is a diagonal matrix its kernel is the product of the annihilators of the diagonal elements, i.e.,

$$\ker(D) = \prod_{j=1}^{r} \{\tilde{y}_j; e_j \circ \tilde{y}_j = 0\}.$$

Summing up the preceding theorems leads to the following algorithm to compute the impulsive behavior of a given IO-system.

**Algorithm 1.** *We start with the system's transfer matrix $H \in \mathbb{C}(s)^{p\times m}$.*

1. *Consider the matrix $H \in \mathbb{C}(s)^{p\times m} = \mathbb{C}(s^{-1})^{p\times m}$ as a matrix of rational functions in $s^{-1}$.*
2. *Compute the matrices $A$ and $B$ from Corollary 5 by*
   (a) *Compute the least common multiple $d$ of the denominators of the entries $H_{ij} = \frac{p_{ij}}{q_{ij}} \in \mathbb{C}(s^{-1})$ of $H$, i.e.,*
   $$d := \mathrm{lcm}_{i,j}(q_{ij})$$
   (b) *We multiply the matrix $H$ by $d$ to eliminate the denominators, i.e.,*
   $$\tilde{H} = dH \in \mathbb{C}[s^{-1}]$$
   (c) *Compute the Smith-Form $D = U\binom{\tilde{H}}{d\mathrm{id}_m}V$ of $\binom{\tilde{H}}{d\mathrm{id}_m}V$. We define the matrix $R$ by*
   $$R = (U_{j-})_{j=p+1,\dots,p+m}.$$
   *We define the matrices $A$ and $B$ by*
   $$A = (R_{-j})_{j=1,\dots,p}, B = (-R_{-j})_{j=p+1,\dots,p+m}.$$
3. *Compute the kernel of the map $A\circ : \Delta^p \to \Delta^p$. We give two possibilities to compute a basis of the kernel. For both we first compute the Smith normal form*
   $$D = UAV = \begin{pmatrix} e_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & e_p \end{pmatrix}$$
   *of the matrix $A$ and compute the prime factorization of $e_p$, i.e.,*
   $$e_p = s^{-\mu(0)} \prod_{\lambda\neq 0} (s^{-1} - \lambda)^{\mu(\lambda)}.$$

- *We compute the representation of $A$ as a polynomial in $s^{-1}$*

$$A = \sum_{k=0}^{n} A_k s^{-k}.$$

*We solve the following system of linear equations*

$$\begin{pmatrix} A_0 & A_1 & \cdots & A_{\mu(0)} \\ 0 & A_0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & A_0 \end{pmatrix} \begin{pmatrix} \tilde{y}_0 \\ \tilde{y}_1 \\ \vdots \\ \tilde{y}_{\mu(0)} \end{pmatrix} = 0.$$

*The kernel* $\ker(A\circ)$ *is generated by the elements*

$$\tilde{y} = \begin{pmatrix} \tilde{y}_0 & \tilde{y}_1 & \cdots & \tilde{y}_{\mu(0)} & 0 & \cdots \end{pmatrix}$$

*where* $\begin{pmatrix} \tilde{y}_0 & \cdots & \tilde{y}_{\mu(0)} \end{pmatrix}^T$ *is a solution of the system of equations above.*

- *The annihilator* $\mathrm{ann}_\Delta(e_i)$ *consists of elements of the form* $\tilde{y}_i := \sum_{j=0}^{\mu_i(0)-1} \lambda_{ij}\delta_j$ *where* $\lambda_{ij} \in \mathbb{C}$*. Hence the kernel of $D$ consists of elements*

$$\tilde{y} = \begin{pmatrix} \tilde{y}_1 \\ \vdots \\ \tilde{y}_p \end{pmatrix} = \begin{pmatrix} \sum\limits_{j=0}^{\mu_1(0)-1} \lambda_{1j}\delta_j \\ \vdots \\ \sum\limits_{j=0}^{\mu_p(0)-1} \lambda_{pj}\delta_j \end{pmatrix}.$$

*Therefore*

$$\ker(A) = \{y = V \circ \tilde{y}; \tilde{y} \in \ker(D)\}.$$

## 5.1 The structure of the impulsive behavior $\mathcal{B}_\infty$

If we consider $\mathcal{B}_\infty$ as behavior itself it is an autonomous discrete behavior since the matrix $A$ has full rank. Therefore its Smith normal form is

$$UAV = \begin{pmatrix} e_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e_p \end{pmatrix} \in \mathbb{C}[s^{-1}]^{p\times m}$$

and consequently the associated module is a torsion module.

Now we want to directly compute the impulsive component $y_0 \in \mathcal{F}_0^p$ of the output $y \in \mathcal{F}_+^p$ from a given input $u \in \mathcal{F}_{+,1}^m$ and a given transfer matrix $H \in \mathbb{C}(s)^{p\times m}$. We decompose the transfer matrix in the following fashion

$$H = H_{\mathrm{pr}} + H_{\mathrm{pol}+}$$

with $H_{\mathrm{pol}+} \in (\mathbb{C}[s]s)^{p\times m}$ of the form

$$H_{\mathrm{pol}+} = \begin{pmatrix} h_{11} & \cdots & h_{1m} \\ \vdots & \ddots & \vdots \\ h_{p1} & \cdots & h_{pm} \end{pmatrix}$$

where the entries

$$h_{ij} = \sum_{k=1}^{n_{ij}} a_{ijk} s^k$$

are polynomials in $s$ of degree $n_{ij}$.

As a preparation we compute the $k^{\text{th}}$ derivative of $Yv$ where $v \in \mathcal{C}^\infty$ is a smooth function.

$$\begin{aligned}
(Yv)^{(k)}(\varphi) &= (-1)^{k-1}(Yv)'(\varphi^{(k-1)}) = (-1)^{k-1}(Y'v)(\varphi^{(k-1)}) + (Yv')(\varphi^{(k-1)}) \\
&= (-1)^{k-1}(\delta v)(\varphi^{(k-1)}) + (-1)^{k-1}(Yv')(\varphi^{(k-1)}) \\
&= (-1)^{k-1}v(0)\varphi^{(k-1)}(0) + (-1)^{k-1}\int_0^\infty v'(t)\varphi^{(k-1)}(t)\,dt \\
&= v(0)\delta^{(k-1)}(\varphi) + (-1)^{k-1}\left(-v'(0)\varphi^{(k-2)}(0) - \int_0^\infty v''(t)\varphi^{(k-2)}(t)\,dt\right) \\
&= v(0)\delta^{(k-1)}(\varphi) + v'(0)\delta^{(k-2)}(\varphi) + (-1)^{k-2}\int_0^\infty v''(t)\varphi^{(k-2)}(t)\,dt
\end{aligned}$$

and by induction

$$(Yv)^{(k)}(\varphi) = \sum_{l=0}^{k-1} v^{(l)}(0)\delta^{(k-1-l)}(\varphi) + (Yu^{(k)})(\varphi).$$

Hence

$$\begin{aligned}
h_{ij} \circ (Yv) &= \sum_{k=1}^{n_{ij}} a_{ijk}s^k \circ (Yv) = \sum_{k=1}^{n_{ij}} a_{ijk}\left(\sum_{l=0}^{k-1} v^{(l)}(0)\delta^{(k-1-l)} + Yu^{(k)}\right) \\
&= \sum_{k=1}^{n_{ij}}\left(\sum_{l=0}^{k-1} a_{ijk}v^{(l)}(0)\delta^{(k-l-1)} + a_{ijk}Yu^{(k)}\right).
\end{aligned}$$

Therefore the $i^{\text{th}}$ component of the impulsive output $(H_{\text{pol}+} \circ u)_0$ corresponding to the input $u \in \mathcal{F}^m_{+,1}$ with $u_j = v_jY$ is

$$\left((H_{\text{pol}+} \circ u)_0\right)_i = \sum_{j=1}^{m}\sum_{k=1}^{n_{ij}}\sum_{l=0}^{k-1} a_{ijk}v_j^{(l)}(0)\delta^{(k-l-1)}.$$

**Remark 7.** Let $H$ be a transfer matrix as above, $n = \max_{i,j}(\deg(H_{ij}))$ and $y = H \circ u$. The impulsive part $y_0$ of the output $y$ depends on the initial values $u(0), u'(0), \dots, u^{(n)}(0)$ of the input $u$ only.

# 6 Extensions: The modules $\mathcal{F}$ and $\mathcal{G}$

In this section we introduce the modules $\mathcal{F}$ and $\mathcal{G}$ which are injective cogenerators. They contain as submodules $\mathcal{F}_+$ resp. $\mathcal{G}_+$. We show that the theory developed for the impulsive solutions of behaviors in $\mathcal{F}_+$ still holds for behaviors in $\mathcal{F}$.

**Theorem and Definition 13.** *The module*

$${}_{\mathbb{C}[s]}\mathcal{F} := \mathcal{C}^\infty + \mathcal{C}^\infty Y + \mathbb{C}[s] \circ \delta$$

*is an injective cogenerator.*

*Proof.* 1. First we show that $\mathcal{F}$ is divisible.

(a) We show that the application of $s$, i.e., the map $s\circ : \mathcal{F} \to \mathcal{F}$, is surjective. We construct the inverse images of all elements of the terms of the sum of $\mathcal{F}$.

i. For every function $u \in \mathcal{C}^\infty$ we define

$$y(t) := \int_0^t u(\tau)\, d\tau \in \mathcal{C}^\infty$$

hence $s \circ y = u$.

ii. For every $u \in \mathcal{C}^\infty Y$ we define

$$y := s^{-1} \circ u$$

since $\mathcal{C}^\infty Y$ is a $\mathbb{C}[s^-1]$-module and $\mathcal{F}_+$ is a $\mathbb{C}(s)$-vector space.

iii. For every distribution $u = \sum_{i=0}^{\infty} \alpha_i \delta^{(i)} \in [s] \circ \delta$ we define

$$y := \alpha_0 Y + \sum_{i=0}^{\infty} \alpha_{i+1} \delta^{(i)} \in \mathcal{F}.$$

Hence $s \circ y = u$.

Therefore the map $s\circ : \mathcal{F} \to \mathcal{F}$ is surjective.

(b) For every $\lambda \in \mathbb{C}$ the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "(s-\lambda)\circ"] & \mathcal{F} \\ \mathcal{F} \arrow[u, "\cdot e^{\lambda t}", "\cong"'] \arrow[r, "s\circ", "\cong"'] & \mathcal{F} \arrow[u, "\cdot e^{\lambda t}", "\cong"'] \end{tikzcd}$$

is commutative and consequently $(s - \lambda)\circ : \mathcal{F} \to \mathcal{F}$ is surjective.

(c) As every polynomial $f \in \mathbb{C}[s]$ is the product of linear factors $f\circ : \mathcal{F} \to \mathcal{F}$ is surjective too.

2. $\mathcal{F}$ is injective since $\mathcal{F}$ is divisible and $\mathbb{C}[s]$ is a principal ideal ring.

3. $\mathcal{C}^\infty$ is an injective cogenerator and $\mathcal{C}^\infty \subseteq \mathcal{F}$ is a submodule of $\mathcal{F}$. Therefore $\mathcal{F}$ is an injective cogenerator.

□

**Remark 8.** The module $\mathcal{F} \subseteq \mathcal{D}'$ is a $\mathbb{C}[s]$- and a $\mathcal{C}^\infty$-submodule of $\mathcal{D}'$.

**Theorem 14.** *The $\mathbb{C}[s]$-module $\mathcal{F} = (\mathcal{C}^\infty + \mathcal{C}^\infty Y) \oplus \mathbb{C}[s] \circ \delta$ is the direct sum of $\mathcal{C}^\infty + \mathcal{C}^\infty Y$ and $\mathbb{C}[s] \circ \delta$, i.e.,*

$$(\mathcal{C}^\infty + \mathcal{C}^\infty Y) \cap (\mathbb{C}[s] \circ \delta) = 0.$$

*Proof.* Let $u \in (\mathcal{C}^\infty + \mathcal{C}^\infty Y) \cap (\mathbb{C}[s] \circ \delta)$ be an element of the intersection. Consequently $u \in (\mathcal{C}^\infty + \mathcal{C}^\infty Y)$ and $u = u_1 + u_2 Y$ where $u_1, u_2 \in \mathcal{C}^\infty$ are smooth functions. We assume that $u \neq 0$, i.e., there exists $0 \neq t_0 \in \mathbb{R}$ where $u(t_0) \neq 0$. Without loss of generality we assume that $u(t_0) > 0$. There exists an interval $I = (t_0 - \varepsilon, t_0 + \varepsilon)$ where $0 \notin I$ and $u(t) > 0$ for $t \in I$ since $u \in \mathcal{C}(\mathbb{R}\backslash\{0\})$.
We choose $\varphi \in \mathcal{D}$ where $\text{supp}(\varphi) = [t_0 - \varepsilon, t_0 + \varepsilon]$ and $\varphi \geq 0$. The evaluation $u(\varphi) = 0$ is zero since $u \in \mathbb{C}[s] \circ \delta$ is a distribution with support contained in zero and $0 \notin \text{supp}(\varphi)$. Hence

$$\begin{aligned} 0 &= \int_{\mathbb{R}} u(t)\varphi(t)\, dt = \int_{\mathbb{R}} (u_1(t) + u_2(t)Y(t))\varphi(t)\, dt \\ &= \int_{-\infty}^{0} u_1(t)\varphi(t)\, dt + \int_0^\infty (u_1(t) + u_2(t))\,\varphi(t)\, dt \end{aligned}$$

There are two cases:

1. $t_0 < 0$:

$$\int_{-\infty}^{0} u_1(t)\varphi(t)\,dt = \int_{t_0-\varepsilon}^{t_0+\varepsilon} \underbrace{u_1(t)\varphi(t)}_{>0}\,dt > 0$$

$$\int_{0}^{\infty} (u_1(t)+u_2(t))\underbrace{\varphi(t)}_{=0}\,dt = 0$$

Therefore $u(\varphi) > 0$, a contradiction.

2. $t_0 > 0$:

$$\int_{0}^{\infty} (u_1(t)+u_2(t))\varphi(t)\,dt = \int_{t_0-\varepsilon}^{t_0+\varepsilon} \underbrace{(u_1(t)+u_2(t))\,\varphi(t)}_{>0}\,dt > 0$$

$$\int_{-\infty}^{0} u_1(t)\underbrace{\varphi(t)}_{=0}\,dt = 0$$

Consequently $u(\varphi) > 0$, a contradiction.

Therefore $u = 0$ and $\mathcal{F} = (\mathcal{C}^\infty + \mathcal{C}^\infty Y) \oplus \mathbb{C}[s] \circ \delta$. □

**Theorem 15** (Borel)**.** *For every given real sequence $(a_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ exists a function $f \in \mathcal{C}^\infty$ where $\frac{f^{(n)}(0)}{n!} = a_n$ for all $n \in \mathbb{N}$.*

*Proof.* The idea is to construct $f$ as a power series $f = \sum_{n\in\mathbb{N}} a_n x^n$ but in general this series is not convergent.
We choose

$$\phi \in \mathcal{D} \text{ where } \phi(x) = \begin{cases} 1 & \text{for} \quad |x| \le \frac{1}{2} \\ 0 & \text{for} \quad |x| \ge 1 \end{cases}$$

Our aim is to choose a sequence $(t_n)_{n\in\mathbb{N}}$ so that for all $m \in \mathbb{N}$ the series

$$\sum_{n=0}^{\infty} a_n (x^n \phi(t_n x))^{(m)}$$

is uniformly convergent on $\mathbb{R}$ and hence

$$\frac{d^{m+1}}{dx^{m+1}} \sum_{n=0}^{\infty} a_n x^n \phi(t_n x) = \sum_{n=0}^{\infty} a_n (x^n \phi(t_n x))^{(m+1)}.$$

In order to achieve this we have to dominate

$$\sum_{n=0}^{\infty} |a_n||(x^n \Phi(t_n x))^{(m)}|.$$

For all $m \in \mathbb{N}$ it is sufficient to dominate

$$\sum_{n=m+1}^{\infty} |a_n||(x^n \Phi(t_n x))^{(m)}|$$

since the first $m+1$ terms of the series are not important for the convergence as $m$ is finite. We define $\Phi_n$ by

$$a_n (x^n \Phi(t_n x))^{(m)} = a_n t_n^{-n} (\underbrace{(t_n x)^n \Phi(t_n x)}_{=:\Phi_n(t_n x)})^{(m)}.$$

Since $\Phi_n(x) = x^n\Phi(x) \in \mathcal{D}$ there exists

$$M_n := \max\left\{\left|\Phi_n^{(m)}\right|; x \in \mathbb{R}, m \in \mathbb{N}, \text{ where } m < n\right\}$$

Hence for all $m, n \in \mathbb{N}$ where $m < n$ holds $\left|\Phi_n^{(m)}(x)\right| \leq M_n$. Consequently for $m < n$

$$\left|(\Phi_n(t_n x))^{(m)}\right| = \left|t_n^m \Phi_n^{(m)}(t_n x)\right| \leq t_n^m M_n.$$

Hence

$$\left|a_n(x^n\Phi(t_n x))^{(m)}\right| \leq |a_n|\, t_n^{-n} t_n^m M_n = |a_n|\, t_n^{-(n-m)} M_n$$

Therefore for all $m \in \mathbb{N}$ the series $\sum_{n=m+1}^{\infty} a_n(x^n\Phi(t_n x))^{(m)}$ is absolutely dominated by

$$\sum_{n=m+1}^{\infty} |a_n|\, M_n t_n^{-(n-m)}. \tag{1}$$

Now we have to choose $t_n$ so that the series (1) is convergent.
We choose a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ where $\varepsilon_n > 0$ and the series $\sum_{n\in\mathbb{N}} \varepsilon_n < \infty$ is convergent. We define

$$t_n := \max\left\{1, \frac{|a_n|\, M_n}{\varepsilon_n}\right\},$$

i.e, $t_n \geq 1$ and therefore $t_n^{-(n-m)} \leq t_n^{-1}$.

$$|a_n|\, M_n t_n^{-(n-m)} \leq |a_n|\, M_n t_n^{-1} \leq |a_n|\, M_n \frac{\varepsilon_n}{|a_n|\, M_n} \leq \varepsilon_n$$

Consequently the series (1) is convergent and therefore for all $m \in \mathbb{N}$ the series

$$\sum_{n=0}^{\infty} a_n(x^n\phi(t_n x))^{(m)}$$

is uniformly convergent on $\mathbb{R}$.
We define

$$f(x) := \sum_{n=0}^{\infty} a_n x^n \Phi(t_n x) \in \mathcal{C}^\infty$$

hence

$$f^{(m)}(x) = \sum_{n=0}^{\infty} a_n(x^n\Phi(t_n x))^{(m)}$$

as the series $f^{(m-1)}(x)$ is uniformly convergent. Consequently

$$f^{(m)}(0) = \sum_{n=0}^{\infty} a_n(x^n\Phi(t_n x))^{(m)}(0).$$

By definition $\Phi(x) = 1$ for $|x| \leq \frac{1}{2}$ therefore $\Phi(t_n x) = 1$ for $x$ close to 0. Consequently

$$(x^n\Phi(t_n x))^{(m)}(0) = (x^n)^m(0) = m!\delta_{n,m}$$

as in the proof of Theorem 7. Therefore $f^m(0) = m!\, a_m$ as claimed. □

**Theorem 16.** *$\mathcal{C}^\infty + \mathcal{C}^\infty Y = \mathcal{C}^\infty(1-Y) + \mathcal{C}^\infty Y$ and*

$$\mathcal{C}^\infty + \mathcal{C}^\infty Y = \left\{u \in \mathcal{C}^\infty(\mathbb{R}\backslash\{0\}); \forall i \in \mathbb{N} : \exists u^{(i)}(0+), \exists u^{(i)}(0-)\right\}$$

*where $u^{(i)}(0+) := \lim_{t\searrow 0} u^{(i)}(t)$ and $u^{(i)}(0-) := \lim_{t\nearrow 0} u^{(i)}(t)$.*

*Proof.* 1. First we show that $\mathcal{C}^\infty + \mathcal{C}^\infty Y = \mathcal{C}^\infty(1-Y) + \mathcal{C}^\infty Y$

$\subseteq$: Let $u \in \mathcal{C}^\infty + \mathcal{C}^\infty Y$. Then $u$ is of the form

$$\begin{aligned} u &= u_1 + u_2 Y = u_1(1 - Y + Y) + u_2 Y \\ &= u_1(1-Y) + (u_1 + u_2)Y \in \mathcal{C}^\infty(1-Y) + \mathcal{C}^\infty Y \end{aligned}$$

$\supseteq$: Let $u \in \mathcal{C}^\infty(1-Y) + \mathcal{C}^\infty Y$. Then $u$ is of the form

$$u = u_1(1-Y) + u_2 Y = u_1 + (u_2 - u_1)Y \in \mathcal{C}^\infty + \mathcal{C}^\infty Y$$

2. Now we show that

$$\mathcal{C}^\infty Y = \left\{ u \in \mathcal{C}^\infty(\mathbb{R}\backslash\{0\}); u(t) = 0 \text{ for } t < 0, \forall i \in \mathbb{N}\, \exists u^{(i)}(0+) \right\}.$$

$\subseteq$: Let $u \in \mathcal{C}^\infty$, i.e., $uY \in \mathcal{C}^\infty Y$.

$$\lim_{t \searrow 0} (uY)^{(i)}(t) = \lim_{t \searrow 0} u^{(i)}(t) = u^{(i)}(0),$$

i.e., $(uY)^{(i)}(0+)$ exists and $(uY)(t) = 0$ for $t < 0$.

$\supseteq$: Let $u \in \mathcal{C}^\infty(\mathbb{R}\backslash\{0\})$ where $u(t) = 0$ for $t < 0$ and there exists $u^{(k)}(0+)$ for all $k \in \mathbb{N}$. We consider the sequence $\left(\frac{u^{(k)}(0+)}{k!}\right)_{k\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$. According to the Theorem of Borel there exists $f \in \mathcal{C}^\infty$ where $\frac{f^k(0)}{k!} = \frac{u^{(k)}(0+)}{k!}$, i.e., $f^{(k)}(0) = u^{(k)}(0+)$. We define

$$v(t) := \begin{cases} f(t) & t \leq 0 \\ u(t) & t > 0 \end{cases}$$

As $v \in \mathcal{C}^\infty(\mathbb{R}\backslash\{0\})$ there exists $v'(t)$ for $t \neq 0$. The fundamental theorem of differential and integral calculus provides

$$v(t) - v(\varepsilon) = \int_\varepsilon^t v'(x)\, dx \text{ for } 0 < \varepsilon < t.$$

Hence

$$v(t) - v(0) = \int_0^t v'(x)\, dx$$

and therefore by the mean-value theorem exists $0 < \tau < t$ such that

$$\frac{v(t) - v(0)}{t} = v'(\tau).$$

Hence $\lim_{t\to 0+} \frac{v(t)-v(0)}{t} = v'(0)$. Consequently $v$ is right-differentiable at zero and

$$v'(0) = \lim_{t\searrow 0} v'(t) = \lim_{t \searrow 0} u'(t) = u'(0+) = \lim_{t \nearrow 0} f'(t) = \lim_{t\nearrow 0} v'(t)$$

hence $v' \in \mathcal{C}^1$ and by induction $v \in \mathcal{C}^\infty$. Consequently $vY \in \mathcal{C}^\infty Y$ and $(vY)(t) = u(t)$ for $t \neq 0$, hence $u \in \mathcal{C}^\infty Y$. Analogously the reader can check for himself that

$$\mathcal{C}^\infty(1-Y) = \left\{ u \in \mathcal{C}^\infty(\mathbb{R}\backslash\{0\}); u(t) = 0 \text{ for } t > 0, \forall k \in \mathbb{N}\, \exists u^{(k)}(0-) \right\}.$$

Therefore

$$\mathcal{C}^\infty + \mathcal{C}^\infty Y = \left\{ u \in \mathcal{C}^\infty(\mathbb{R}\backslash\{0\}); \forall i \in \mathbb{N} : \exists u^{(i)}(0+), \exists u^{(i)}(0-) \right\}.$$

$\square$

We define $\mathcal{F}_1 := \mathcal{C}^\infty + \mathcal{C}^\infty Y$ and $\Phi = \text{proj} : \mathcal{F} \to \Delta$. Then the following sequences are exact.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_1 \arrow[r, hook] & \mathcal{F} \arrow[r, "\Phi"] & \Delta \arrow[r] & 0 \\
0 \arrow[r] \arrow[u, hook] & \mathcal{F}_{+,1} \arrow[r, hook] \arrow[u, hook] & \mathcal{F}_+ \arrow[r, "\Phi"] \arrow[u, hook] & \Delta \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

**Theorem 17.** *The torsion submodule of the $\mathbb{C}[s]$-module $\mathcal{D}'$ of distributions equals the torsion submodule of the $\mathbb{C}[s]$-module of $\mathcal{C}^\infty$, i.e.,*

$$\mathrm{T}(\mathcal{D}') = \mathrm{T}(\mathcal{C}^\infty) = \oplus_{\lambda \in \mathbb{C}} \mathbb{C}[t] e^{\lambda t}$$

*Proof.* 1. First we show by induction that the annihilator $\text{ann}_{\mathcal{D}'}(s^\mu) = \mathbb{C}[t]_{<\mu}$.

$\mu = 1$: Let $T \in \text{ann}_{\mathcal{D}'}(s)$ be distribution where $s \circ T = T' = 0$. Hence the evaluation on all functions $\varphi \in \mathcal{D}$ is zero, i.e.,

$$0 = T(\varphi) = -T(\varphi').$$

Consequently the application of $T$ onto elements of $\mathcal{H} = \text{im}(s\circ : \mathcal{D} \to \mathcal{D}) = \{\psi \in \mathcal{D}; \exists \varphi \in \mathcal{D} : \varphi' = \psi\}$ is zero, i.e.,

$$\forall \psi \in \mathcal{H} : T(\psi) = 0.$$

Now we show that $\mathcal{H}$ equals the hyperplane $\ker\left(\mathcal{D} \to \mathbb{C}, \varphi \mapsto \int\limits_{\mathbb{R}} \varphi(\tau)\, d\tau\right)$, i.e. $\mathcal{H} = \left\{\varphi \in \mathcal{D}; \int\limits_{\mathbb{R}} \varphi(\tau)\, d\tau = 0\right\}$.

$\subseteq$: Let $\psi \in \mathcal{H}$ and $\varphi \in \mathcal{D}$ where $\psi' = \varphi$ hence

$$\varphi(t) = \int_{-\infty}^{t} \psi(\tau)\, d\tau.$$

Since $\varphi \in \mathcal{D}$ there exists a real number $N \in \mathbb{R}$ where $\varphi(t) = 0$ for $t \geq N$. Therefore $\int_{\mathbb{R}} \psi(\tau)\, d\tau = 0$.

$\supseteq$: Let $\varphi \in \mathcal{D}$ be a test function where $\int_{\mathbb{R}} \varphi(\tau)\, d\tau$. We define a function by

$$\psi(t) := \int_{-\infty}^{t} \varphi(\tau)\, d\tau.$$

$\psi$ is in $\mathcal{C}^\infty$ since $\varphi \in \mathcal{D} \subset \mathcal{C}^\infty$ is a test function. We have to show that $\psi$ has a compact support. There exists a real number $N \in \mathbb{R}$ so that support $\text{supp}(\varphi) \subset [-N, N]$ since $\varphi \in \mathcal{D}$. Let $t \leq -N$ hence

$$\psi(t) = \int_{-\infty}^{t} \underbrace{\varphi(\tau)}_{=0}\, d\tau = 0.$$

If $t \geq N$ then

$$0 = \int_{-\infty}^{\infty} \varphi(\tau)\, d\tau = \underbrace{\int_{-\infty}^{t} \varphi(\tau)\, d\tau}_{=\psi(t)} + \underbrace{\int_{t}^{\infty} \varphi(\tau)\, d\tau}_{=0}$$

Consequently $\psi(t) = 0$ for $t \notin [-N, N]$ and therefore $\varphi \in \mathcal{D}$.

The hyperplane $\mathcal{H}$ is closed, since it is the kernel of the distribution 1. We choose a test function $\varphi_0 \in \mathcal{D}$ where $\int_{\mathbb{R}} \varphi(\tau)\, d\tau = 1$. Let $\varphi \in \mathcal{D}$ be an arbitrary test function. We define $\lambda := \int_{\mathbb{R}} \varphi(\tau)\, d\tau$ then $\varphi$ has the representation

$$\varphi = \underbrace{(\varphi - \lambda\varphi_0)}_{=:\psi\in\mathcal{H}} + \lambda\varphi_0.$$

Application of the distribution $T$ on $\varphi$ leads to

$$T(\varphi) = \underbrace{T(\psi)}_{=0} + \lambda T(\varphi_0) = T(\varphi_0)1(\varphi).$$

Therefore $T = \underbrace{T(\varphi_0)}_{\in\mathbb{C}} 1 \in \mathbb{C}$.

$\mu - 1 \to \mu$: We consider the equation

$$0 = s^\mu \circ T = s \circ (s^{\mu-1} \circ T).$$

Hence $s^{\mu-1} \circ T = c \in \mathbb{C}$ and

$$0 = s^{\mu-1} \circ T - c = s^{\mu-1} \circ (T - c\frac{t^{\mu-1}}{(\mu-1)!}).$$

The distribution $T - c\frac{t^{\mu-1}}{(\mu-1)!} = p(t)$ is by assumption a polynomial of degree smaller or equal than $\mu-2$. Hence $T = p(t) + c\frac{t^{\mu-1}}{(\mu-1)!}$ is a polynomial of degree smaller or equal than $\mu - 1$.

2. To show that $\mathrm{T}_{s-\lambda}(\mathcal{D}') = \mathbb{C}[t]^{\lambda t}$ it is sufficient to show that

$$\mathrm{T}_{s-\lambda}(\mathcal{D}') \subseteq \mathrm{T}_{s-\lambda}(\mathcal{C}^\infty).$$

Let $u \in \mathcal{D}'$ we define $v := ue^{-\lambda t}$. As a preparation we compute

$$(s - \lambda) \circ v = u'e^{-\lambda t} + \lambda ue^{-\lambda t} - \lambda ue^{-\lambda t} = u'e^{-\lambda t},$$

hence by induction

$$(s - \lambda)^\mu \circ v = u^{(\mu)}e^{-\lambda t}.$$

Let $v$ an element of the annihilator $\mathrm{ann}_{\mathcal{D}'}((s - \lambda)^\mu)$ then

$$0 = (s - \lambda)^\mu \circ v = u^{(\mu)}e^{-\lambda t}$$

and therefore $u^{(\mu)} = 0$. Consequently $u$ is a polynomial of degree smaller than $\mu$ and $v = ue^{\lambda t} \in \mathbb{C}[t]e^{\lambda t}$.

□

We consider the IO system

$$\mathcal{B}_{\mathcal{F}} := \left\{ \begin{pmatrix} y \\ u \end{pmatrix} \in \mathcal{F}^{p+m}; P \circ y = Q \circ u \right\}$$

where $(P, -Q) \in \mathbb{C}[s]^{k\times(p+m)}$ is a matrix of polynomials in $s$ with $\mathrm{rank}(P, -Q) = \mathrm{rank}(P) = p$. Furthermore $PH = Q$ and $H \in \mathbb{C}(s)^{p\times m}$ is a $p$ times $m$ matrix of polynomials in $s$.

**Remark 9.** The behavior $\mathcal{B}_{\mathcal{F}_+}$ is the intersection of $\mathcal{B}_{\mathcal{F}}$ and $\mathcal{F}_+^{p+m}$, i.e.,

$$\mathcal{B}_{\mathcal{F}_+} = \left\{ \begin{pmatrix} y \\ u \end{pmatrix} \in \mathcal{F}_+^{p+m}; P \circ y = Q \circ u \text{ resp. } y = H \circ u \right\} = \mathcal{B}_{\mathcal{F}} \cap \mathcal{F}_+^{p+m}.$$

**Theorem 18** (Impulsive Solutions)**.** *Let $u \in \mathcal{F}_1^m = (\mathcal{C}^\infty)^m + (\mathcal{C}^\infty Y)^m$ be a function that is smooth on $\mathbb{R} \setminus \{0\}$ and possibly has a discontinuity at zero. We decompose it into the sum $u = u_1 + u_2$ where $u_1$ is a vector of smooth functions and $u_2 \in \mathcal{F}_{+,1}^m$.*

1. *Let $y \in \mathcal{F}^p$ a solution of the equation $P \circ y = Q \circ u$ where $u$ is given. Then its impulsive part is the image of $u_2$ by $H$, i.e.,*

$$\Phi(y) = \Phi\left((H \circ u_2)_0\right).$$

2. *The impulsive behavior $\mathcal{B}_\infty$ of $H$ is the image of the set of outputs $y$ under $\Phi$ where the inputs $u$ are elements of $\mathcal{F}_1^m$, i.e.,*

$$\Phi(H \circ \mathcal{F}_{+,1}^m) = \left\{ \Phi(y); \begin{pmatrix} y \\ u \end{pmatrix} \in \mathcal{B}_\mathcal{F}, u \in \mathcal{F}_1^m \right\}.$$

*Proof.* Since $\mathcal{C}^\infty$ is $\mathbb{C}[s]$-injective there exists a vector of smooth functions $y_1 \in (\mathcal{C}^\infty)^p$ where

$$P \circ y_1 = Q \circ u_1.$$

Let $y_2$ be the image of $u_2$ by the transfer matrix $H$, i.e.,

$$y_2 = H \circ u_2 \in \mathcal{F}_+^p = (\mathcal{C}^\infty)^p \oplus (\mathbb{C}[s] \circ \delta)^p,$$

hence $P \circ y_2 = Q \circ u_2$ and $\binom{y_2}{u_2} \in \mathcal{B}_{\mathcal{F}_+}$. Therefore the sum $\binom{y_1+y_2}{u_1+u_2}$ is an element of $\mathcal{B}_\mathcal{F}$, i.e.,

$$P \circ (y_1 + y_2) = Q \circ u_1 + Q \circ u_2 = Q \circ u.$$

Let $y \in \mathcal{F}^p$ with $P \circ y = Q \circ u$. Using $y_1$ and $y_2$ from above we obtain

$$P \circ (y - (y_1 + y_2)) = P \circ y - P \circ (y_1 + y_2) = P \circ y - Q \circ u = 0.$$

Let

$$\mathcal{B}^0 = \{z \in \mathcal{F}^p; P \circ z = 0\} = \{z \in (\mathcal{C}^\infty)^p; P \circ z = 0\}$$

be the autonomous part of $\mathcal{B}$. It is a submodule of the torsion module $\mathrm{T}(\mathcal{D}')$ of the module of distributions. According to Theorem 17 it consists of polynomial exponential functions and therefore

$$\Phi(y) = \Phi(\underbrace{y_1}_{\in \mathcal{C}^\infty}) + (\underbrace{y - (y_1 + y_2)}_{\in \mathrm{T}(\mathcal{C}^\infty)} + y_2) = \Phi((H \circ u_2)_1 + (H \circ u_2)_0) = \Phi((H \circ u_2)_0)$$

□

We have seen that the impulsive solutions in $\mathcal{F}^p$ depend only on the elements $u_2 \in (\mathcal{C}^\infty Y)^m$. As this part is in $\mathcal{F}_+^p$ the impulsive behavior corresponding to $\mathcal{B}_\mathcal{F}$ is the same as the impulsive behavior corresponding to $\mathcal{B} \subset \mathcal{F}_+^p$. Therefore the computation of the impulsive behavior remains the same. $\mathcal{F}$ is the smallest $\mathbb{C}[s]$-submodule of $\mathcal{D}'$ that contains $\mathcal{C}^\infty$ and the Heaviside function $Y$. As we have seen in t contains all functions $u \in \mathcal{C}^\infty(\mathbb{R} \setminus \{0\})$ such that all its derivatives at $t = 0$ are left- and right-continuous.
Technically we can consider electrical networks, that are switched on at time $t = 0$. Another interpretation is that at time $t = 0$ the system changes because of the failure of one of its components.
As we have considered both systems in $\mathcal{F}$ and $\mathcal{F}_+$ we have given two possibilities to consider systems which abruptly "change" at a specific time $t = 0$. Compared with each other, they have the following advantages resp. disadvantages:

The advantage of $\mathcal{F}$ compared to $\mathcal{F}_+$ is that $\mathcal{F}$ is an injective cogenerator and $\mathcal{F}_+$ is not.
The advantage of $\mathcal{F}_+$ compared to $\mathcal{F}$ is that $\mathcal{F}_+$ is a $\mathbb{C}(s)$-vector space and the subbehavior of $\mathcal{F}_+$ are subspaces.

# 7 Examples

At the beginning of this chapter we introduce the concept of Electrical Networks. Then we consider examples from electrical engineering and compute their impulsive behaviors.

**Definition 13.** A finite graph $\Gamma$ is given by

- a finite set $V$ of vertices
- a finite set $K$ of edges
- two maps $\mathrm{dom}, \mathrm{cod} : K \to V, k \mapsto \mathrm{dom}(k), \mathrm{cod}(k)$.

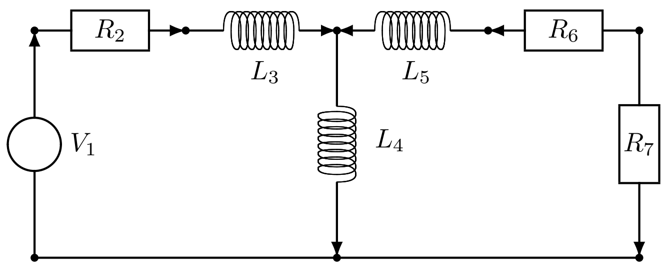
The vertex $v := \mathrm{dom}(k)$ of an edge $k \in K$ is called the initial vertex of $k$ and the vertex $w := \mathrm{cod}(k)$ is called its final vertex. We shortly write $\Gamma = (V, K)$ where we assume that the maps dom resp. cod are clear from the context.

**Definition 14** (Electrical Network)**.** An electrical network is a connected digraph $\Gamma = (V, K)$ where for every edge $k \in K \uplus (-K)$ there are two functions, the voltage $V_k \in \mathscr{F}$ and the current $I_k \in \mathscr{F}$, which fulfill the following conditions:

1. The voltage and current functions of the inverse edges are the negative functions of the original edge, i.e.,
$$V_{-k} = -V_k \text{ and } I_{-k} = -I_k.$$
2. Kirchhoff's circuit laws:
   (a) For every cycle $\omega = (k_1, \dots, k_n) \subseteq K \uplus (-K)$ the sum of the potential differences is zero, i.e.,
   $$\sum_{i=1}^{n} V_{k_i} = 0.$$
   (b) For every vertex $v \in V$ the sum of the current flowing into $v$ equals the sum of the current that flows out of $v$, i.e.,
   $$\sum_{\{k \in K; \mathrm{dom}(k)=v\}} I_k - \sum_{\{k \in K; \mathrm{cod}(k)=v\}} I_k = 0$$
3. For every edge $k \in E$ there is either a relation between $V_k$ and $I_k$ or one of the functions is given. There are three possible relations:
   (a) If $k$ is a resistor edge then
   $$V_k = R_k I_k$$
   where $R_k \in \mathbb{R}_+$ is a positive real.
   (b) If $k$ is a capacitor edge then
   $$I_k = C_k \dot{V}_k = C_k(s \circ V_k).$$
   where $C_k \in \mathbb{R}_+$ is a positive real.
   (c) If $k$ is an inductor edge then
   $$V_k = L_k \dot{I}_k = L_k(s \circ I_k)$$
   where $L_k \in \mathbb{R}_+$ is a positive real.

**Example 4.** The circuit for this example is taken from [2], p. 359. We consider a transformer equivalent circuit with respect to the stray electrostatic field. The quotient $\frac{N_1}{N_2}$ is the relation between the winding factors of the primary and the secondary coil. The constant $L_{\sigma_1}$ resp. $L_{\sigma_2}$ is the primary resp. the secondary leakage inductance. We use the following shortened notation

$$L_5 = \left(\frac{N_1}{N_2}\right)^2 L_{\sigma_2} \text{ and } L_3 = L_{\sigma_1}.$$




We assume that the resistor $R_7$ stands for a device that needs a specific voltage and want to compute the therefore needed input voltage $V_1$.
Kirchhoff's circuit laws lead to the following equations

$$\begin{aligned}
V_1 + V_2 + V_3 + V_4 &= 0 \\
V_4 + V_5 + V_6 - V_7 &= 0 \\
-I_1 + I_2 &= 0 \\
-I_2 + I_3 &= 0 \\
-I_3 + I_4 - I_5 &= 0 \\
-I_1 + I_4 + I_7 &= 0 \\
-I_5 + I_6 &= 0.
\end{aligned}$$

The equations of the resistor edges are

$$\begin{aligned}
-V_2 + R_2 I_2 &= 0 \\
-V_6 + R_6 I_6 &= 0 \\
-V_7 + R_7 I_7 &= 0.
\end{aligned}$$

The inductor edges lead to the equations

$$\begin{aligned}
-V_3 + L_3 s I_3 &= 0 \\
-V_4 + L_4 s I_4 &= 0 \\
-V_5 + L_5 s I_5 &= 0.
\end{aligned}$$

Hence the system is

$$\mathcal{B} = \{z \in \mathcal{F}_+; R \circ z = 0\}$$

where the matrix $R$ is

$$R = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & R_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & L_3s & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & L_4s & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & L_5s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & R_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & R_7 \end{pmatrix}$$

and

$$z = (V_1,\, I_1,\, V_2,\, I_2,\, V_3,\, I_3,\, V_4,\, I_4,\, V_5,\, I_5,\, V_6,\, I_6,\, V_7,\, I_7)^T .$$

The manifest variables are $V_7$ and $V_1$ since we want to compute $V_1$ for a given $V_7$. We eliminate the latent variables $I_1, \ldots, I_7, V_2, \ldots V_6$ by calculating a left annihilator to $R_{\text{lat}} = (R_{-2} \cdots R_{-12}, R_{-14})$. The resulting system for the manifest variables is

$$\left\{ \begin{pmatrix} V_1 \\ V_7 \end{pmatrix} \in \mathcal{F}_+^2; R_{\text{new}} \begin{pmatrix} V_1 \\ V_7 \end{pmatrix} = 0 \right\}$$

where

$$R_{\text{new}} = \left( R_7L_4s,\, a_2s^2 + a_1s + R_7R_2 + R_2R_6 \right)$$

and

$$a_2 = L_5L_4 + L_3L_5 + L_4L_3,\; a_1 = R_7L_4 + R_7L_3 + R_6L_4 + R_2L_5 + L_3R_6 + R_2L_4.$$

We want to choose the voltage at edge 7 ($V_7$) and calculate the therefore needed voltage $V_1$. This leads to the IO system

$$(R_{\text{new}})_{-1} \circ V_1 = (R_{\text{new}})_{-2} \circ V_7.$$

The transfer function is

$$h = -\frac{(L_5L_4+L_3L_5+L_4L_3)s^2+(R_7L_4+R_7L_3+R_6L_4+R_2L_5+L_3R_6+R_2L_4)s+R_7R_2+R_2R_6}{R_7L_4s}.$$

Hence the matrix $A$ as in Corollary 5 is

$$A = \begin{pmatrix} R_7L_4s^{-1} \end{pmatrix}.$$

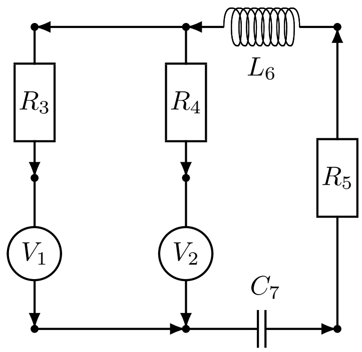
To calculate the kernel of $A\circ : \Delta \to \Delta$ we have to solve the following system of linear equations:

$$\begin{pmatrix} 0 & R_7L_4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0.$$

The solution is $b = \mathbb{C}\binom{1}{0}$ and hence $\ker(A\circ) = \mathbb{C}\delta_0$. Therefore the impulsive behavior of $h$ is

$$\mathcal{B}_\infty = \Phi(h \circ \mathcal{F}_+) = \mathbb{C}\delta_0.$$

**Example 5.** We consider the following electric circuit.




Kirchhoff's circuit laws lead to the following equations

$$\begin{aligned}
V_1 + V_3 + V_5 + V_6 + V_7 &= 0 \\
V_2 + V_4 + V_5 + V_6 + V_7 &= 0 \\
I_1 + I_2 - I_7 &= 0 \\
I_1 - I_2 &= 0 \\
I_3 - I_4 &= 0 \\
I_5 - I_7 &= 0 \\
I_3 + I_4 - I_6 &= 0.
\end{aligned}$$

The equations of the resistor edges are

$$\begin{aligned}
-V_3 + R_3 I_3 &= 0 \\
-V_4 + R_4 I_4 &= 0 \\
-V_5 + R_5 I_5 &= 0.
\end{aligned}$$

The inductor edge leads to the equation

$$-V_6 + L_5 s I_6 = 0.$$

The capacitor edge leads to the equation

$$C_7 s V_7 - I_7 = 0.$$

Hence the system is

$$\mathcal{B} = \{z \in \mathcal{F}_+; R \circ w = 0\}$$

where the matrix $R$ is

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & R_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & R_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & R_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & L_5 s & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_7 s & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

and

$$w = (V_1,\, I_1,\, V_2,\, I_2,\, V_3,\, I_3,\, V_4,\, I_4,\, V_5,\, I_5,\, V_6,\, I_6,\, V_7,\, I_7)^T\,.$$

We choose $V_1$ and $V_2$ as input and $V_5$, $I_5$ as output. Hence the manifest variables are $V_1$, $V_2$, $V_5$ and $I_5$. We eliminate the other variables by computing a left annihilator $X$ for the matrix

$$R_{\mathrm{lat}} = (R_{-2}, R_{-4}, \ldots, R_{-8}, R_{-11}, \ldots, R_{-14})\,.$$

The resulting new system matrix is

$$\begin{aligned} R_{\mathrm{new}} &= X R_{\mathrm{man}} = X\,(R_{-1}, R_{-3}, R_{-9}, R_{-10}) = \\ &= \begin{pmatrix} -1 & R_5 & 0 & 0 \\ C_7 s R_4 - C_7 s R_3 & -R_3 + R_4 & C_7 s\,(R_4 + 2\,L_5 s) & -C_7 s\,(2\,L_5 s + W_3) \end{pmatrix} \end{aligned}$$

The IO system with input $(V_1, V_2)$ and output $(V_5, I_5)$ as chosen above is

$$P \circ \begin{pmatrix} V_5 \\ I_5 \end{pmatrix} = Q \circ \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}.$$

where

$$P = \begin{pmatrix} -1 & R_5 \\ C_7 s R_4 - C_7 s R_3 & -R_3 + R_4 \end{pmatrix}$$

and

$$Q = \begin{pmatrix} 0 & 0 \\ -C_7 s\,(R_4 + 2\,L_5 s) & C_7 s\,(2\,L_5 s + R_3) \end{pmatrix}.$$

The transfer matrix is

$$H = \begin{pmatrix} \frac{R_5 C_7 s (R_4 + 2\,L_5 s)}{R_3 - R_4 - R_5 C_7 s R_4 + R_5 C_7 s R_3} & -\frac{R_5 C_7 s (2\,L_5 s + R_3)}{R_3 - R_4 - R_5 C_7 s R_4 + R_5 C_7 s R_3} \\ \frac{C_7 s (R_4 + 2\,L_5 s)}{R_3 - R_4 - R_5 C_7 s R_4 + R_5 C_7 s R_3} & -\frac{C_7 s (2\,L_5 s + R_3)}{R_3 - R_4 - R_5 C_7 s R_4 + R_5 C_7 s R_3} \end{pmatrix}.$$

Hence the matrix $A$ as in Corollary 5 is

$$A = \begin{pmatrix} 0 & -\frac{(R_3 - R_4)^2 s^{-1} \left(s^{-1} + C_7 R_5\right)}{C_7 R_4} \\ -C_7 R_4 & R_5 C_7 R_4 \end{pmatrix}.$$

To calculate the kernel of $A\circ : \Delta \to \Delta$ we have to solve the following system of linear equations:

$$\begin{pmatrix} 0 & 0 & 0 & -\frac{(R_3-R_4)^2R_5}{R_4} & 0 & -\frac{(R_3-R_4)^2}{CR_4} \\ -C_7R_4 & R_5C_7R_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{(R_3-R_4)^2R_5}{R_4} \\ 0 & 0 & -C_7R_4 & R_5C_7R_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -C_7R_4 & R_5C_7R_4 \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_6 \end{pmatrix} = 0.$$

The solution is $b = \mathbb{C}(R_5, 1, 0, 0, 0, 0)^T$ and hence $\ker(A\circ) = \mathbb{C}\binom{R_5\delta_0}{\delta_0}$. Therefore the impulsive behavior of $H$ is

$$\mathcal{B}_\infty = \Phi(H \circ \mathcal{F}_+) = \mathbb{C}\begin{pmatrix} R_5\delta_0 \\ \delta_0 \end{pmatrix}.$$

We choose $R_5 = 0$ what technically means causing a short circuit. Since the output is $\binom{V_5}{I_5}$, we expect that the first component of the impulsive behavior becomes zero and the second component becomes "worse", i.e. a multiple of $\delta_1$, corresponding to $\delta'$, appears. Actually the impulsive behavior becomes

$$\overline{\mathcal{B}}_\infty = \Phi(\bar{H} \circ \mathcal{F}_+) = \mathbb{C}\begin{pmatrix} 0 \\ \delta_0 \end{pmatrix} + \mathbb{C}\begin{pmatrix} 0 \\ \delta_1 \end{pmatrix}.$$

# A Maple procedures

Enclosed to this master's thesis is a CD with a Maple file with procedures to compute the impulsive solutions and the impulsive behavior of a given transfer matrix. The examples given in the last section can also be found on the CD. In this section we describe the most important procedures and their syntax.

For computational reasons the $\delta$-distribution is encoded as the indeterminate $d$ and its $k$-th derivative $\delta^{(k)}$ is encoded as $d^{k+1}$.

**ApplyTransferMatrix** This function computes the output of a system from a given input, i.e., it computes $y = H \circ u$ for given $H$ and $u$.

**Syntax.** ApplyTransferMatrix(H,u), where $H$ is a matrix of polynomials in $s$ and $u$ is a vector of functions in $t$.

**PolPPart** The function PolPPart computes the polynomial part of a matrix of rational functions, i.e., for a given matrix $A \in \mathbb{C}(s)^{m\times n}$ it computes $A_{\text{pol}+} \in (\mathbb{C}[s]s)^{m\times n}$.

**Syntax.** PolPPart(A), where $A$ is a matrix of rational functions in $s$.

**PrPart** The function PrPart computes the proper part of a matrix of rational functions, i.e., for a given matrix $A \in \mathbb{C}(s)^{m\times n}$ it computes $A_{pr} \in (\mathbb{C}(s)_{\text{pr}})^{m\times n}$.

**Syntax.** PrPart(A), where $A$ is a matrix of rational functions in $s$.

**Phi** This function computes the projection of a given vector $y \in \mathcal{F}^p$ resp. $y \in \mathcal{F}_+^p$ onto $\Delta^p$.

**Syntax.** Phi(y), where $y$ is a vector of functions (or distributions) in $t$.

**Convertsi** This function converts a matrix of rational functions in $s$ into matrix of rational functions in $s^{-1} = r$.

**Syntax.** Convertsi(A), where $A$ is a matrix of rational functions in $s$.

**UCR** This function computes the matrices $A$ and $B$ from Corollary 5.

**Syntax.** UCR(H), where $H$ is a matrix of rational functions in $r = s^{-1}$.

**SolveDiscrete1** and **SolveDiscrete2** These functions compute the basis of the kernel of a matrix of polynomials in $s^{-1}$ in $\Delta^p$, i.e., they compute the solutions of a system of difference equations. They use the two methods given in Algorithm 1.

**Syntax.** SolveDiscrete1(A) resp. SolveDiscrete2(A), where $A$ is a matrix of polynomials in $r = s^{-1}$.

**ImpulsivePart** The function ImpulsivePart directly computes the impulsive part of the output $y$ for a given input $u$ and a given transfer matrix $H$.

**Syntax.** ImpulsivePart(H,u), where $H$ is a matrix of rational functions in $s$ and $u$ a vector of functions in $t$.

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