The Method of Alternating (Linear) Projections in Finite Dimensional Banach Spaces

Bachelor thesis topic

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In the context of the development of the theory of linear operators on Hilbert spaces, J. von Neumann showed in [3] that given two closed subspaces $M, N \subset H$ of a Hilbert space H with orthogonal projections P_M and P_N, the sequence

$$x_0 := x, \qquad x_{2n+1} := P_M x_{2n} \quad \text{and} \quad x_{2n} := P_N x_{2n-1} \tag{1}$$

converges to $P_{M\cap N}(x)$ for all $x \in H$. More precisely, he showed that the sequence of the iterates of $P_N P_M$ converges to the orthogonal projection onto $M \cap N$ in the strong operator topology. Algorithm (1) is called the *von Neumann alternating projection algorithm*.

Outside Hilbert spaces the situation is much more complicated. First of all there is no immediate concept of orthogonality and the mapping which assigns to each point of the space the point of a given subspace with minimal distance need not be linear. Even worse, there may be spaces which do not admit any linear projection onto them at all. On the positive side, in [1], R. E. Bruck and S. Reich showed that in uniformly convex Banach spaces, the sequence in (1) converges when P_M and P_N are projections of norm one.

A different approach to the question of whether the sequence in (1) converges has been approached in [2] where the concept of an angle between two projections (instead of between subspaces) has been introduced.

The aim of this thesis is to study the behaviour of the alternating projection method for linear projections in finite dimensional subspaces.

References

[1] Ronald E. Bruck and Simeon Reich. Nonexpansive projections and resolvents of accretive operators in Banach spaces. *Houston J. Math.*, 3(4):459–470, 1977.

[2] Izhar Oppenheim. Angle criteria for uniform convergence of averaged projections and cyclic or random products of projections. Preprint (arXiv:1605.00220), 2016.

[3] John von Neumann. On rings of operators. Reduction theory. *Ann. of Math. (2)*, 50:401–485, 1949.