Method of infinite system of equations for problems in unbounded domains

Dang Quang A

Institute of Information Technology
18 Hoang Quoc Viet, Cau giay, Hanoi, Vietnam

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Outline

• Many problems of mechanics and physics are posed in unbounded domains: heat transport problems in infinite or semi-infinite bar, aerosol propagation in atmosphere, problem of ocean pollution, ...

• For solving these problems one usually restricts oneself to treat the problem in a bounded domain and try to use available efficient methods for finding exact or approximate solution in the restricted domain.

Questions: which size of restricted domain is enough and how to set conditions on artificial boundary for obtaining approximate solution with good accuracy? .

1. Simplest way: to transfer boundary condition on infinity to the artificial boundary.
2. Set appropriate conditions on artificial boundary (Dang, Tsynkov, Colonius,Halpern,...).
3. Set transparent conditions on artificial boundary (Arnold, Ehrhardt, Schulte, Sofronov, ...)

• Quasi-uniform grid for mapping unbounded domain to bounded one (Alshin, Alshina, Boltnev, Kalitkin).
• We approach to problems in unbounded domain by infinite system of linear equations.:

- construct difference scheme for the problem in unbounded domain
- suggest a method for treating the infinite system in order to obtain an approximate solution with a given accuracy.

This infinite system approach overcomes drawbacks of the quasi-uniform grid method.

Infinite system of equations (Kantorovich and Krylov)

- Canonical form:

$$x_i = \sum_{k=1}^{\infty} c_{ik} x_k + b_i, \quad i = 1, 2, \ldots \tag{1}$$

- The solution found by the method of successive approximation

$$x_i^{(n+1)} = \sum_{k=1}^{\infty} c_{ik} x_k^{(n)} + b_i, \quad i = 1, 2, \ldots; n = 1, 2, \ldots$$

with the zero starting approximation $x_i^{(0)} = 0 \quad (i = 1, 2, \ldots)$ is called the main solution of the system.

- Set

$$\rho_i = 1 - \sum_{k=1}^{\infty} |c_{ik}|.$$

- The system (1) is called regular system if

$$\rho_i > 0, \quad i = 1, 2, ...;$$

it is called completely regular if there exists a constant $\theta > 0$ such that

$$\rho_i \geq \theta, \quad i = 1, 2, ...$$

- Assume that there exists a number K such that the free members b_i satisfy the condition

$$|b_i| \leq K\rho_i, \quad i = 1, 2, ... \tag{2}$$

Theorem

The regular system (1) with the free members satisfying the condition (2) has a bounded solution $|x_i| \leq K$ which can be found by the method of successive approximation.

Theorem

The main solution of the regular system (1) with the free members satisfying the condition (2) can be found by the truncation method, that is, if x_i^N *is the solution of the finite system*

$$x_i = \sum_{k=1}^{N} c_{ik} x_k + b_i, \quad i = 1, 2, ..., N$$

then

$$x_i^* = \lim_{N \to \infty} x_i^N,$$

where x_i^* *(i=1, 2,...) is the main solution of the system.*

Quasi-uniform grid (Alshin, Alshina, Kalitkin and Panchenko, 2002)

- Let $x(\xi)$ be strictly monotone smooth function of the argument $\xi \in [0,1]$. The grid

$$\omega_N = \{x_i = x(i/N), \quad i = 0, 1, ..., N\}$$

with $x(0) = a, \; x(1) = +\infty$ is called quasi-uniform grid on $[a, +\infty]$. In this case the last node x_N of the grid is on the infinity.

- Example of quasi-uniform grids are the grids

$$\omega_N = \{x_i = \frac{i}{N-i}, \quad i = 0, 1, ..., N\} \quad \text{(hyperbolic grid)},$$

$$\omega_N = \{x_i = \tan\frac{\pi i}{2N}, \quad i = 0, 1, ..., N\} \quad \text{(tangential grid)}.$$

- The model problem of heat conductivity in a semi-infinite bar

$$\begin{gathered} -(ku')' + du = f(x), \quad x > 0, \\ u(0) = \mu_0, \quad u(+\infty) = 0 \end{gathered} \tag{3}$$

- Difference scheme on the uniform grid $\{x_i = ih, \ i = 0, 1, ...\}$

$$-\frac{1}{h}\left(a_{i+1}\frac{y_{i+1} - y_i}{h} - a_i\frac{y_i - y_{i-1}}{h}\right) + d_i y_i = f_i, \ i = 1, 2, ...$$
$$y_0 = \mu_0, \ y_i \to 0, \ i \to \infty,$$

$$a_i = k(x_i - h/2), \ d_i = d(x_i), \ f_i = f(x_i).$$

- Canonical form of infinite system

$$\begin{aligned} & y_i = p_i y_{i-1} + q_i y_{i+1} + r_i, \;\; i = 0, 1, 2, ... \\ & y_i \to 0, \;\; i \to \infty. \end{aligned} \tag{4}$$

- Progonka method for tridiagonal system of equations:

$$y_i = \alpha_{i+1} y_{i+1} + \beta_{i+1}, \;\; i = 0, 1, ..., \tag{5}$$

where coefficients are calculated by the formulas:

$$\begin{aligned} & \alpha_1 = 0, \;\; \beta_1 = \mu_0, \\ & \alpha_{i+1} = \frac{q_i}{1 - p_i \alpha_i}\ , \;\; \beta_{i+1} = \frac{r_i + p_i \beta_i}{1 - p_i \alpha_i}, \;\; i = 1, 2, ... \end{aligned} \tag{6}$$

Theorem

Given an accuracy $\varepsilon > 0$. *If starting from a natural number* N *there holds*

$$\frac{|\beta_i|}{1-\alpha_i} \le \varepsilon, \ \forall i \ge N+1 \tag{7}$$

then for the deviation of the solution of the truncated system

$$\begin{aligned} \bar{y}_i &= p_i \bar{y}_{i-1} + q_i \bar{y}_{i+1} + r_i, \ \ i = 0, 1, ..., N, \\ \bar{y}_i &= 0, \ \ i \ge N+1 \end{aligned} \tag{8}$$

compared with the solution of the infinite system (4) there holds the following estimate

$$\sup_i |y_i - \bar{y}_i| \le \varepsilon. \tag{9}$$

Example

$$-((1+\frac{1}{1+x})u')' + (1+\sin^2 x)u = f(x)$$

$$u(0)=1,\ u(+\infty)=0,$$

Exact solution $u(x) = \frac{1}{1+x^2}$.

Table: Case $h = 0.1$

ε	N	error
0.01	108	0.0085
0.001	325	0.0012

Parabolic equation on semi-infinite bar

$$\begin{aligned} &\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},\ 0 < x < +\infty,\ \ t > 0, \\ &u(x,0) = 0,\ \ u(0,t) = 1,\ \ u(+\infty,t) = 0. \end{aligned} \tag{10}$$

- Exact solution

$$u(x,t) = \frac{2}{\sqrt{\pi}} \int_{x/2\sqrt{kt}}^{+\infty} \exp(-\xi^2) d\xi.$$

- Infinite system on each time layer $j+1$:

$$\begin{aligned} &-ry_{i-1}^{j+1} + (1+2r)y_i^{j+1} - ry_{i+1}^{j+1} = y_i^j,\ \ i = 1,2,\dots \\ &y_0^{j+1} = 1,\ \ y_i^{j+1} \to 0,\ \ i \to \infty, \end{aligned} \tag{11}$$

where $r = k\tau/h^2$

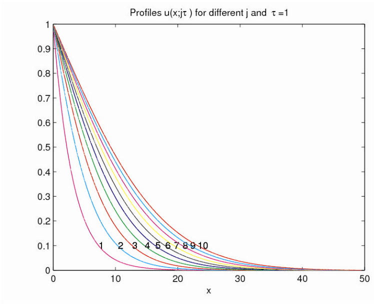

Figure: By infinite system

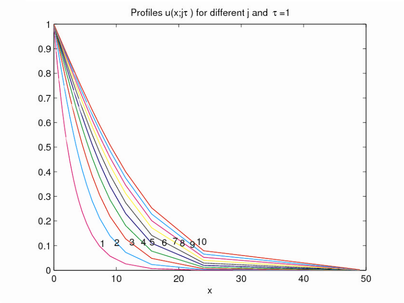

Figure: By quasi-uniform grid

Stationary problem of air pollution

$$u\frac{\partial\varphi}{\partial x} - w_g\frac{\partial\varphi}{\partial z} - \frac{\partial}{\partial x}\nu\frac{\partial\varphi}{\partial z} + \sigma\varphi = 0,\ x > 0$$

$$u\varphi = Q\delta(z - H),\ x = 0$$

$$\frac{\partial\varphi}{\partial z} = \alpha\varphi,\ z = 0,\ \varphi \to 0,\ z \to \infty,$$

The numerical solution on uniform grid using the infinite system method was studied by Dang (1994), where a theorem similar to the above Theorem was proved.

Problem describing ion wave in stratified incompressible fluid

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 u}{\partial x^2} - u\right) + \frac{\partial^2 u}{\partial x^2} = 0, \;\; x > 0, \;\; t > 0,$$
$$u(0, t) = f(t), \;\; u(+\infty, t) = 0,$$
$$u(x, 0) = f_1(x), \;\; \frac{\partial u}{\partial t}(x, 0) = 0.$$

Set $\phi = \frac{\partial^2 u}{\partial x^2} - u$. Then the problem is decomposed into

$$\begin{cases} \frac{\partial^2 \phi}{\partial t^2} + \phi + u = 0, \\ \phi(x, 0) = f_1''(x) - f_1(x), \end{cases} \qquad \begin{cases} \frac{\partial \phi}{\partial t}(x, 0) = 0, \\ \frac{\partial^2 u}{\partial x^2} - u = \phi(x, t), \\ u(0, t) = f(t), \;\; u(+\infty, t) = 0. \end{cases}$$

- Difference schemes

$$\frac{\phi_i^{j+1} - 2\phi_i^j + \phi_i^{j-1}}{\tau^2} + \phi_i^j + u_i^j = 0,\ j = 1, 2, \dots \tag{12}$$
$$\phi_i^0 = f_1''(x_i) - f_1(x_i),\ \phi_i^1 = \phi_i^0.$$

$$-\frac{u_{i-1}^{j+1} - 2u_i^{j+1} + u_{i+1}^{j+1}}{h^2} + u_i^{j+1} = -\phi_i^{j+1}, \tag{13}$$
$$u_0^{j+1} = f^{j+1},\ u_i^{j+1} \to 0,\ i \to \infty.$$

Example 1: Take initial conditions be homogeneous and the left boundary condition $u(0,t) = \arctan^2(10t).\sin(0.3t)$.

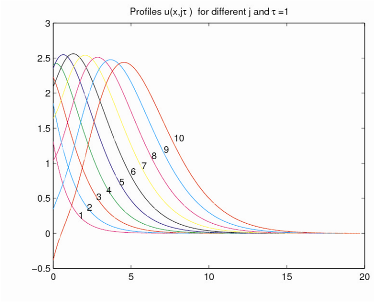

Figure: By infinite system

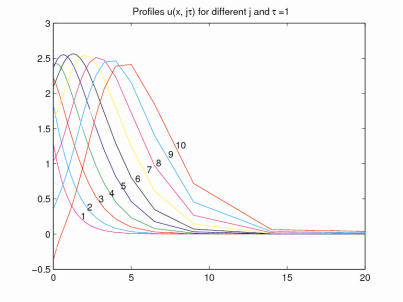

Figure: By quasi-uniform grid

Example 2: The left boundary condition is zero, the initial condition is

$$u(x,0) = x^3 e^{-x}, \ \frac{\partial u}{\partial t}(x,0) = 0.$$

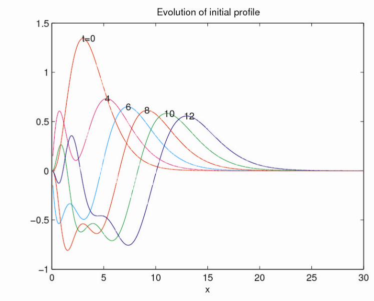

Figure: By infinite system

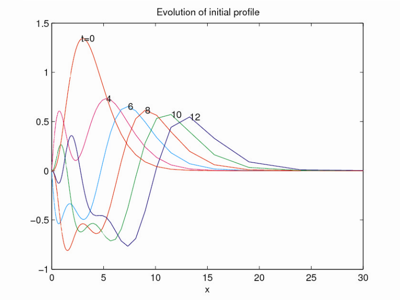

Figure: By quasi-uniform grid

Summary and outlook

• Propose and investigate the infinite system method for solving several one-dimensional stationary and nonstationary problems, where the keystone is when truncate the infinite system.

• This method reveals advantage over the quasi-uniform grid method in time-dependent problems, especially in problems of wave propagation.

• In combination with the alternating directions method the method can be applied to two-dimensional problems in semi-infinite and infinite strips?

THANKS FOR ATTENTION!