The Collocation Method and the Truncation-of-Fourier-Series Method in the numerical solution of Boundary Value Problems for Functional Differential Equations

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Outline

1. Motivation
   - The problem
   - Previous Work and aim of the research

2. Our Contribution
   - The approach for the numerical solution
   - The methods
   - Convergence results

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Non-neutral Functional Differential Equation BVPs

- The problem:

$$\begin{cases} y'(t) = F(t, y), \ t \in [a, b], \\ B(y) = 0, \end{cases}$$

where

  - $F : [a, b] \times V \to \mathbb{R}^d$,
  - $B : V \to \mathbb{R}^d$,
  - V is the Banach space of the continuous functions $[a, b] \to \mathbb{R}^d$.

Neutral Functional Differential Equation BVPs

- The problem:

$$\begin{cases} y'(t) = F(t, y, y'), \ t \in [a, b], \\ B(y, y') = 0, \end{cases}$$

where

- $F : [a, b] \times V \times U \to \mathbb{R}^d$,
- $B : V \times U \to \mathbb{R}^d$,
- U is a suitable Banach space of functions $[a, b] \to \mathbb{R}^d$.

Particular cases of FDEs

- The form

$$y'(t) = F(t, y, y'),\ t \in [a, b], \tag{1}$$

encompass

  - integro-differential equations

$$y'(t) = f\left(t, y(t), \int_{\theta_1(t)}^{\theta_2(t)} k(t, s, y(s), y'(s))\, ds\right),\ t \in [a, b].$$

  - differential equations with deviating arguments

$$\begin{aligned} y'(t) = f\,(t, y(t), y(\theta_1(t)), \ldots, y(\theta_k(t)),\\ y'(\vartheta_1(t)), \ldots, y'(\vartheta_\kappa(t))),\\ t \in [a, b]. \end{aligned}$$

The side condition

- The form (1) requires
  - $\theta_1(t), \theta_2(t) \in [a, b]$ for Fredholm integro-differential equations;
  - $\theta_i(t), \vartheta_r(t), \in [a, b]$, $i = 1, \ldots, k$ and $r = 1, \ldots, \kappa$, for differential equations with deviating arguments.
- If this is not the case, we have to use the side condition

$$y(t) = \phi(t) \text{ and } y'(t) = \varphi(t), \quad t < a \text{ or } t > b.$$

The side condition

- We regain the form (1) by inserting the side condition in the definition of F.

- Integro-differential equations are rewritten as

$$y'(t) = f\left(t, y(t), \int_{\theta_1(t)}^{\theta_2(t)} K\left(t, s, y, y'\right) ds\right), \quad t \in [a, b],$$

where

$$K(t, s, y, y') = \begin{cases} k(t, s, y(s), y'(s)) & \text{if } s \in [a, b] \\ k(t, s, \phi(s), \varphi(s)) & \text{if } s < a \text{ or } s > b. \end{cases}$$

The side condition

- Differential equations with deviating arguments are rewritten as

$$y'(t) = f\left(t, y(t), \Theta_1(t,y), \ldots, \Theta_k(t,y), \mathit{\Theta}_1(t,y'), \ldots, \mathit{\Theta}_\kappa(t,y')\right), \quad t \in [a,b].$$

where

$$\Theta_i(t,y) = \begin{cases} y(\theta_i(t)) \text{ if } s \in [a,b] \\ \phi(\theta_i(t)) \text{ if } s < a \text{ or } s > b \end{cases}, \quad i = 1, \ldots, k$$

$$\mathit{\Theta}_r(t,y') = \begin{cases} y'(\vartheta_r(t)) \text{ if } s \in [a,b] \\ \varphi(\vartheta_r(t)) \text{ if } s < a \text{ or } s > b \end{cases}, \quad r = 1, \ldots, \kappa.$$

The side condition

- Observation: the side condition is not considered as a boundary condition.

Particular cases of boundary conditions

- The form
$$B(y) = 0$$
encompass
  - Classical boundary condition: $g(y(a), y(b)) = 0$. In particular:
    - periodic boundary condition: $y(a) = y(b)$;
    - for second order equation:
      Dirichlet boundary conditions: $y(a) = y_{0a}$, $y(b) = y_{0b}$;
      Neumann boundary conditions: $y'(a) = y_{1a}$, $y'(b) = y_{1b}$;
      Robin boundary conditions:
      $c_0 \cdot (y(a), y(b)) - c_1 \cdot (y'(a), -y'(b)) = (d_a, d_b)$.

Particular cases of boundary conditions

- Non-local boundary conditions for second order equation:
  - non-local Dirichlet boundary conditions: $y(a) = G_{0a}(y)$, $y(b) = G_{0b}(y)$;
  - non-local Neumann boundary conditions: $y'(a) = G_{1a}(y)$, $y'(b) = G_{1b}(y)$.
- Nicoletti-type boundary conditions:

$$g(y(a), y(b), y(t_1), \ldots, y(t_m)) = 0,$$

  where $t_i \in [a, b]$, $i = 1, \ldots, m$.
- integral boundary conditions

$$g\left(y(a), y(b), \int_a^b w(s, y(s))ds\right) = 0.$$

Neutral BVPs with unknown parameters

- The problem:

$$\begin{cases} y'(t) = F(t, y, y', p), \ t \in [a, b], \\ B(y, y', p) = (0, 0), \end{cases}$$

  where

  - $p \in \mathbb{R}^{d_0}$ is a vector of unknown parameters,
  - $F : [a, b] \times V \times U \times \mathbb{R}^{d_0} \to \mathbb{R}^d$,
  - $B : V \times U \times \mathbb{R}^{d_0} \to \mathbb{R}^d \times \mathbb{R}^{d_0}$.

- Motivating example: the computation of periodic solution for autonomous FDEs reduces to a FDE BVP containing the period of the periodic solution as an unknown parameter.

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Numerical literature

- Many papers dealing with BVPs for integro-differential equations.
  - A selected example: R. J. Hangelbroek, H. G. Kaper and G. K. Leaf. Collocation methods for integro-differential equations. Vol. 14 Siam J. Numer. Anal. (1977).
    - BVP:
$$\begin{cases} y'(t) = f(t, y(t)) + \int_a^b K(t, s, y(s), y'(s))ds, \; t \in [a, b], \\ By = c. \end{cases}$$
    - Method: collocation on ν nodes.
    - Convergence results: uniform order ν; Legendre collocation nodes $\Rightarrow$ uniform order $\nu + 1$ and nodal order 2ν.

Numerical literature

- Many papers dealing with non-neutral BVPs for differential equations with deviated arguments.
  - A Selected example: G. Bader (1985). Solving Boundary Value Problems for Functional Differential Equations by Collocation. Progress in Scientific Computing Vol. 5. Numerical Boundary Value ODEs (1985).
    - BVP:

$$\begin{cases} y'(t) = f(t, y(t), y(\theta_1(t)), \ldots, y(\theta_k(t))), \ t \in [a, b], \\ By = c. \end{cases}$$

    - Method: collocation on ν nodes.
    - Convergence results: uniform order ν; no superconvergenge results.

Numerical literature

- Only two papers dealing with neutral BVPs for differential equations with deviated arguments.
  - D. A. W. Barton, B. Krauskopf and R. E. Wilson. Collocation schemes for periodic solutions of neutral delay differential equations. Journal of Difference Equations and Applications, 12 (2006).
    - BVP: computation of periodic solution of $y'(t) = f(y(t), y(t-\tau), y'(t-\tau))$ of unknow period T reduces to

$$\begin{cases} z'(s) = Tf(z(s), (Pz)(s-\frac{\tau}{T}), \frac{1}{T}(Pz)'(s-\frac{\tau}{T})), \ s \in [0,1], \\ z(0) = z(1) \\ b(z,T) = 0 \quad \leftarrow \textit{phase condition}. \end{cases}$$

      where Pz is the prolongation by periodicity of z to $\mathbb{R}$.
    - Method: collocation on ν nodes.
    - Convergence results: experimental work, no proof of convergence.

Numerical literature

- K. Engelborghs and E. J. Doedel. Convergence of a boundary value difference equation for computing periodic solutions of neutral delay differential equations. Journal of Difference Equations and Applications, 7 (2001), 927-940.
  - BVP: computation of periodic solution of $y'(t) + dy'(t-\tau)) = a(t)y(t) + b(t)y(t-\tau) + f(t)$ of know period 1 reduces to

$$\begin{cases} z'(s) + d(Pz)'(s-\tau) = a(s)z(s) + b(s)(PZ)(s-\tau) + f(s) \\ \hfill s \in [0,1] \\ z(0) = z(1). \end{cases}$$

  - Method: forward finite differences approximations for the derivatives.
  - Convergence results: order one.

Numerical literature

- Only two papers dealing with BVPs for general second order non-neutral FDEs.
  - C. W. Cryer. The numerical solution of boundary value problem for second order functional differential equations by finite differences. Numerische Mathematik 20 (1972/73).
    - BVP:

$$\begin{cases} y''(t) = f(t, y(t)) + \mathcal{F}(y)(t), \ t \in [0,1], \\ y(0) = y_0, y(1) = y_1. \end{cases}$$

    - Method: central finite difference scheme.
    - Convergence results: order two.

Numerical literature

- T. Jankowski. Boundary value problem for systems of functional differential equations. Applications of Mathematics 47 (2002)
  - BVP:
    $$\begin{cases} y''(t) = \mathcal{F}(y, y')(t), \ t \in [a, b] \\ y(a) = y_a, y(b) = y_b. \end{cases}$$
  - Method: special continuous method for second order equations.
  - Results: arbitrarily high order of convergence if $\mathcal{F}(y, y')$ does not depend on y'; order two if $\mathcal{F}(y, y')$ depends on y'.

Aim of the research

- Study the numerical computation of solutions of the problem

$$\begin{cases} y'(t) = F(t, y, y', p), \; t \in [a, b], \\ B(y, y', p) = 0. \end{cases} \tag{2}$$

- There are no papers dealing with such a general form of BVPs, even in the non-neutral case.
- We introduce a general type of discretization including:
  - the collocation method,
  - the truncation-of-Fourier series method.

Outline

Reformulation of the BVP

- BVP:

$$\begin{cases} y'(t) = F(t, y, y', p), \ t \in [a, b], \\ B(y, y', p) = 0. \end{cases}$$

- Assumption on $F : [a, b] \times V \times U \times \mathbb{R}^{d_0} \to \mathbb{R}^d$:

$$F(\cdot, v, u, \beta) \in U, \ (v, u, \beta) \in V \times U \times \mathbb{R}^{d_0}.$$

- Hence, we introduce

$$\mathcal{F} : V \times U \times \mathbb{R}^{d_0} \to U, \ \mathcal{F}(v, u, \beta) = F(\cdot, v, u, \beta)$$

  and the BVP becomes

$$\begin{cases} y' = \mathcal{F}(y, y', p) \\ B(y, y', p) = 0. \end{cases}$$

- We use y' as the unknown.

Reformulation of the BVP: the Green operator $\mathcal{G}$

- Let $\mathcal{G} : U \times \mathbb{R}^d \to V$ be a linear bounded operator such that, for any $u \in U$, the differential equation

$$v'(t) = u(t), \ t \in [a,b], \tag{3}$$

of unknown $v \in V$ has the set of solutions

$$\left\{ \mathcal{G}(u,\alpha) : \alpha \in \mathbb{R}^d \right\}.$$

- $\mathcal{G}$ is a Green operator for (3).

- Examples: for $c \in [a,b]$,

$$\mathcal{G}(u,\alpha)(t) = \int_c^t u(s)\,ds + \alpha, \ t \in [a,b].$$

Reformulation of the BVP: the Green operator $\mathcal{G}$

- Our philosophy is to interpretate the differential equation

$$v'(t) = u(t), \ t \in [a, b],$$

as

$$v = \mathcal{G}(u, \alpha) \text{ for some } \alpha \in \mathbb{R}^d.$$

- In other words, we replace

the derivative operator

with

the Green operator.

Reformulation of the BVP: the operator Φ

- The BVP can be restated as the search for zeros of the operator

$$\Phi : X \to X, \quad X := U \times \mathbb{R}^{d} \times \mathbb{R}^{d_0}$$
$$\Phi\left(u, \alpha, \beta\right) = \left(u - \mathcal{F}\left(\mathcal{G}\left(u, \alpha\right), u, \beta\right), B\left(\mathcal{G}\left(u, \alpha\right), u, \beta\right)\right).$$

- We have:

(y, p) is a solution of the BVP

$$\Updownarrow$$

$y = \mathcal{G}\left(u^{*}, \alpha^{*}\right)$ and $p = \beta^{*}$

for some zero $\left(u^{*}, \alpha^{*}, \beta^{*}\right)$ of Φ.

Discretization: $\widehat{U}_K$, π_K, ρ_K and $\widehat{\Phi}_K$

- For any index of discretization K we introduce:
  - a finite dimensional space $\widehat{U}_K$,
  - a linear bounded operator $\pi_K : \widehat{U}_K \to U$ (prolongation),
  - a linear bounded operator $\rho_K : U \to \widehat{U}_K$ (restriction),
  - the operator $\widehat{\Phi}_K : \widehat{X}_K \to \widehat{X}_K$, $\widehat{X}_K := \widehat{U}_K \times \mathbb{R}^d \times \mathbb{R}^{d_0}$,

    $$\widehat{\Phi}_K (u_K, \alpha, \beta) = \left(u_K - \rho_K \mathcal{F}\left(\mathcal{G}\left(\pi_K u_K, \alpha\right), u, \beta\right), B\left(\mathcal{G}\left(\pi_K u_K, \alpha\right), u, \beta\right)\right)$$

    approximation of $\Phi : X \to X$, $X = U \times \mathbb{R}^d \times \mathbb{R}^{d_0}$,

    $$\Phi(u, \alpha, \beta) = \left(u - \mathcal{F}\left(\mathcal{G}(u, \alpha), u, \beta\right), B\left(\mathcal{G}(u, \alpha), u, \beta\right)\right).$$

Discretization: the approach

- Approach for the numerical solution of functional differential equation BVPs: compute the zeros of the finite-dimensional operator $\widehat{\Phi}_K$.

- Given a zero $\left(\widehat{u}_K^*, \alpha_K^*, \beta_K^*\right)$ of $\widehat{\Phi}_K$,
  - $\left(\pi_K \widehat{u}_K^*, \alpha_K^* . \beta_K^*\right)$ is an approximation of a zero of Φ,
  - $\pi_K \widehat{u}_K^*$ is an approximation of y',
  - (v_K^*, β_K^*), where $v_K^* = \mathcal{G}\left(\pi_K \widehat{u}_K^*, \alpha_K^*\right)$, is an approximation of the solution (y, p) of the BVP.

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The methods

- The type of discretization presented includes:
  - Collocation $\rightarrow \left\{ \begin{array}{l} \text{the Finite Element Method (FEM)} \\ \text{the Spectral Element Method (SEM),} \end{array} \right.$
  - Truncation of Fourier Series $\rightarrow$ the Spectral Method (SM).

Discretization: Collocation (d=1)

- The solution y is sufficientely smooth on certain subintervals $I_j = [\xi_{j-1}, \xi_j]$, $j = 1, \ldots, m$, of the domain $[a, b]$, where
$$a = \xi_0 < \xi_1 < \ldots < \xi_{m-1} < \xi_m = b$$

- U is the space of functions $u : [a, b] \to \mathbb{R}$ such that the restrictions $u_I|_{I_j}$, $j = 1, \ldots, m$, are bounded and measurable.

- On any subinterval $I_j = [\xi_{j-1}, \xi_j]$, $j = 1, \ldots, m$, we introduce a mesh:
$$\Omega_j = \left\{ t_{j,0}, t_{j,1}, \ldots, t_{j,N_j-1}, t_{j,N_j} \right\}$$
$$\xi_{j-1} = t_{j,0} < t_{j,1} < \ldots < t_{j,N_j-1} < t_{j,N_j} = \xi_j$$

Discretization: Collocation (d=1)

- Collocation nodes:

$$0 \leq c_1 < \ldots < c_\nu \leq 1.$$

- $\widehat{U}_K$ is the space $\mathbb{R}^{\nu(N_1+\ldots+N_m)}$:

$$\begin{array}{l} \widehat{u} = \left(u_{j,n+1,i}\right) \in \widehat{U}_K, \\ \widehat{u}_{j,n+1,i} \text{ value at } t_{j,n} + c_i h_{j,n+1}, \\ j = 1,\ldots,m,\ \ n = 0,\ldots,N_{K,j} - 1 \text{ and } i = 1,\ldots,\nu_K. \end{array}$$

- Prolongation $\pi_K : \widehat{U}_K \to U$: for $\widehat{u} \in \widehat{U}_K$, $j = 1,\ldots,m$ and $n = 0,\ldots,N_j - 1$ and

$$\left(\pi_K \widehat{u}\right)|_{[t_{j,n},t_{j,n+1}]} = \text{polynomial interpolating the values } \widehat{u}_{j,n+1,i} \text{ at the nodes } t_{j,n} + c_i h_{j,n+1},\ i = 1,\ldots,\nu$$

Discretization: Collocation (d=1)

- Restriction $\rho_K : U \to \widehat{U}_K$: for $u \in U$, $j = 1, \ldots, m$ and $n = 0, \ldots, N_j - 1$

$$(\rho_K u)_{j,n+1,i} = u\left(t_{j,n} + c_i h_{j,n+1}\right), \ i = 1, \ldots, \nu.$$

Discretization: FEM and SEM

- We have two possibilities to obtain better and better approximations:
  - FEM: ν is fixed and $N \to \infty$ as $K \to \infty$,
  - SEM: N is fixed and $\nu \to \infty$ as $K \to \infty$.

Discretization: SM (d=1)

- U is the space of functions $u : [a, b] \to \mathbb{R}$ such that, for any $j = 1, \ldots, m$ and $n = 0, \ldots, N_j - 1$, the function

$$\overline{u|_{[t_{j,n}, t_{j,n+1}]}}(c) = u(t_{j,n} + ch_{j,n+1}), \ c \in [0, 1],$$

belongs to $L^{2,w}([0, 1])$.

- Let $\{\varphi_i\}_{i=1,2,\ldots}$ be an orthonormal basis of $L^{2,w}([0, 1])$.

- Every $f \in L^{2,w}([0, 1])$ is expressed by the Fourier series

$$f = \sum_{i=1}^{\infty} \kappa_i \varphi_i,$$

where

$$\kappa_i = \langle f, \varphi_i \rangle_{L^{2,w}([0,1])}, \ i = 1, 2, \ldots .$$

Discretization: SM (d=1)

- $\widehat{U}_K$ is the space $\mathbb{R}^{\nu(N_1+\ldots+N_m)}$:

$$\widehat{u} = (u_{j,n+1,i}) \in \widehat{U}_K,$$
$$\widehat{u}_{j,n+1,i} \text{ Fourier coefficient in } [t_{j,n}, t_{j,n+1}],$$
$$j = 1, \ldots, m, \; n = 0, \ldots, N_j - 1 \text{ and } i = 1, \ldots, \nu.$$

- Prolongation $\pi_K : \widehat{U}_K \to U$: for $\widehat{u} \in \widehat{U}$, $j = 1, \ldots, m$ and $n = 0, \ldots, N_j - 1$

$$\overline{(\pi_K \widehat{u})|_{[t_{j,n}, t_{j,n+1}]}}(c) = (\pi_K \widehat{u})(t_{j,n} + c h_{j,n}) = \sum_{i=1}^{\nu} \widehat{u}_{j,n+1,i} \phi_i(c), \; c \in [0, 1]$$

Discretization: SM (d=1)

- Restriction $\rho_K : U \to \widehat{U}_K$: for $u \in U$, $j = 1, \ldots, m$ and $n = 0, \ldots, N_j - 1$

  $(\rho_K u)_{j,n+1,i} = i$–th Fourier coefficient of $\overline{u|_{[t_{j,n},t_{j,n+1}]}}$, $i = 1, \ldots, \nu$.

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The error on the solution (y, p)

- Let $x^* = (u^*, \alpha^*, \beta^*)$ be a zero of Φ.
- Under suitable stability conditions, for sufficientely large K, $\widehat{\Phi}_K$ has a unique zero $\widehat{x}_K^* = (\pi_K u_K^*, \alpha_K^*, \beta_K^*)$ in a neighborhood of x^*.
- (v_K^*, β_K^*), where $v_K^* = \mathcal{G}\left(\pi_K u_K^*, \alpha_K^*\right)$, is an approximation of the solution (y, p) of the BVP.
- We are interested in the order, as $K \to \infty$, of the error

$$\|v_K^* - y\|_\infty + \|\beta_K^* - p)\|_\infty .$$

- We assume y sufficientely smooth on each inteval I_j, $j = 1, \ldots, m$.

FEM convergence order

- For general neutral BVPs: order $\mathcal{O}\left(h^{\nu}\right)$, where ν is the number of collocation points.

- For non-neutral BVPs and neutral integro-differential equations BVPs: order $\mathcal{O}\left(h^{\min\{p,\nu+1\}}\right)$, where p is the order of the quadrature rule of nodes $c_1, \ldots, c_\nu$.

- This last result was known in case of

$$y'(t) = f(t, y(t)) + \int_a^b K(t, s, y(s), y'(s))ds, \ \ t \in [a, b],$$

with Legendre collocation points, but not in general. In particular not known for non-neutral differential equations with deviating arguments.

SM convergence order

- Order $\mathcal{O}\left(\frac{1}{\nu^q}\right)$, where ν is the number of Fourier coefficients, whenever y is of class C^q on each interval I_j, $j = 1, \ldots, m$.

- Exponential order $\mathcal{O}\left(\mathrm{e}^{-\gamma\nu}\right)$, whenever y is analytic on each interval I_j, $j = 1, \ldots, m$.

Summary and outlook

- Summary
  - We have considered a general type of discretization for functional differential equation BVPs.
  - Several methods are included: FEM, SEM, SM.

- Outlook
  - Extension to Functional Partial Differential Equations

    $$\begin{cases} Dy(x) = F(x, y), \ x \in \Omega, \\ B(y) = 0 \end{cases}$$

    where D is a differential operator, e.g. $D = \Delta$. The mathematics is the same: now, $\mathcal{G}$ is the Green operator for

    $$(Dv)(x) = u(x), \ x \in \Omega.$$

  - Nodal superconvergence for particular equations.