Backward difference time discretization of parabolic equations on evolving surfaces

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Outline

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Parabolic equations on evolving surfaces

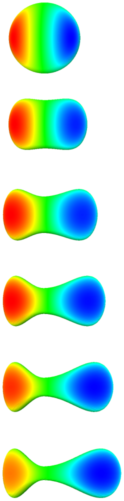
Finite element space discretization

Backward difference time discretization

Stability and error bounds

Numerical experiment

Diffusion on a sphere deforming to a baseball bat



Notation: evolving surface

$\Gamma(t)$, for $t \in [0, T]$, is a smoothly evolving family of smooth, compact d-dimensional hypersurfaces in $\mathbb{R}^{d+1}$

$n = n(x, t)$ normal vector field to $\Gamma(t)$

$v = v(x, t)$ velocity of the surface

Notation: derivatives in time and space

material derivative of a function $u = u(x,t)$:

$$\dot{u} = \frac{\partial u}{\partial t} + v \cdot \nabla u.$$

tangential gradient: projection of the gradient to the tangent space

$$\nabla_\Gamma u = \nabla u - \nabla u \cdot n \; n$$

Laplace-Beltrami operator on Γ is the tangential divergence of the tangential gradient:

$$\Delta_\Gamma u = \nabla_\Gamma \cdot \nabla_\Gamma u.$$

Linear parabolic equation on the evolving surface

conservation of a scalar quantity $u = u(x,t), x \in \Gamma(t), t \in [0,T]$ with a linear diffusive flux on $\Gamma(t)$ is modelled by

$$\dot{u} + u\nabla_\Gamma \cdot v - \Delta_\Gamma u = f \quad \text{on } \Gamma$$

Dziuk & Elliott 2007

Weak formulation

$$\frac{d}{dt}\int_\Gamma u\varphi + \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma \varphi = \int_\Gamma u\dot{\varphi} + \int_\Gamma f\varphi$$

for all smooth $\varphi = \varphi(x,t)$, $x \in \Gamma(t)$, $t \in [0, T]$

obtained via Leibniz formula

$$\frac{d}{dt}\int_\Gamma g = \int_\Gamma \dot{g} + g\nabla_\Gamma \cdot v$$

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Surface discretization

surface triangulation $\Gamma_h(t) = \bigcup_{T(t)\in\mathcal{T}(t)} T(t)$

with simplices $T(t)$ having vertices $a_i(t) \in \Gamma(t)$

piecewise linear nodal basis functions: $\phi_j(a_i(t), t) = \delta_{ji}$

finite element space $S_h(t) = \text{span}\{\phi_1(\cdot, t), \ldots, \phi_N(\cdot, t)\}$

discrete velocity $v_h(x, t) = \sum_{j=1}^{N} v(a_j(t), t)\phi_j(x, t)$

Discrete-surface derivatives

discrete material derivative on the discrete evolving surface:

$$\dot{u}_h = \frac{\partial u_h}{\partial t} + v_h \cdot \nabla u_h$$

key property of basis functions:

$$\dot{\phi}_i = 0$$

discrete surface gradient (understood in a piecewise sense):

$$\nabla_{\Gamma_h} u_h = \nabla u_h - \nabla u_h \cdot n_h \; n_h.$$

Finite element discretization of the parabolic PDE

$$\frac{d}{dt}\int_{\Gamma_h} u_h\varphi_h + \int_{\Gamma_h} \nabla_{\Gamma_h} u_h \cdot \nabla_{\Gamma_h}\varphi_h = \int_{\Gamma_h} u_h\dot{\varphi}_h + \int_{\Gamma_h} f_h\varphi_h \quad \forall \varphi_h \in S_h(t).$$

system of ODEs for nodal values $u(t) = (u_i(t))$:

$$\frac{d}{dt}\big(M(t)u(t)\big) + A(t)u(t) = f(t)$$

with mass and stiffness matrices

$$M(t)_{ij} = \int_{\Gamma_h(t)} \phi_i(\cdot,t)\phi_j(\cdot,t),$$
$$A(t)_{ij} = \int_{\Gamma_h(t)} \nabla_{\Gamma_h(t)}\phi_i(\cdot,t)\cdot\nabla_{\Gamma_h(t)}\phi_j(\cdot,t)$$

Dziuk & Elliott 2007, including error analysis

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BDF

With step size $\tau > 0$, discretize

$$\frac{1}{\tau}\sum_{j=0}^{k} \delta_j M(t_{n-j})u_{n-j} + A(t_n)u_n = f(t_n), \quad n \geq k,$$

for given starting values $u_0, \dots, u_{k-1}$.

Method coefficients δ_j are given by

$$\delta(\zeta) = \sum_{j=0}^{k} \delta_j \zeta^k = \sum_{\ell=1}^{k} \frac{1}{\ell}(1-\zeta)^\ell.$$

Order k, 0-stable for $k \leq 6$.

Lemma from Dahlquist's G-stability theory (1978)

Let $\delta(\zeta)$ and $\mu(\zeta)$ be polynomials of degree at most k. Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathbb{R}^N$ with associated norm $|\cdot|$. If

$$\text{Re}\,\frac{\delta(\zeta)}{\mu(\zeta)} > 0 \quad \text{for} \quad |\zeta| < 1,$$

then there exists a symmetric positive definite matrix $(g_{ij}) \in \mathbb{R}^{k \times k}$ such that for all $v_0, \ldots, v_k \in \mathbb{R}^N$

$$\Big\langle \sum_{i=0}^{k} \delta_i v_{k-i}, \sum_{j=0}^{k} \mu_j v_{k-j} \Big\rangle \geq \sum_{i,j=1}^{k} g_{ij} \langle v_i, v_j \rangle - \sum_{i,j=1}^{k} g_{ij} \langle v_{i-1}, v_{j-1} \rangle.$$

Lemma from Nevanlinna & Odeh (1981)

If $k \leq 5$, then there exists $0 \leq \eta < 1$ such that for
$\delta(\zeta) = \sum_{\ell=1}^{k} \frac{1}{\ell}(1-\zeta)^{\ell}$,

$$\mathrm{Re}\,\frac{\delta(\zeta)}{1-\eta\zeta} > 0 \quad \text{for} \quad |\zeta| < 1.$$

The smallest possible value of η is found to be
$\eta = 0$, 0, 0.0836, 0.2878, 0.8160 for $k = 1, \ldots, 5$.

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Defects and errors

The ODE solution satisfies the BDF relation up to a defect d_n:

$$\frac{1}{\tau}\sum_{j=0}^{k}\delta_j M(t_{n-j})u(t_{n-j}) + A(t_n)u(t_n) = f(t_n) - d_n.$$

For smooth solutions we have by Taylor expansion (in suitable norms!) $d_n = \mathcal{O}(\tau^k)$. The error

$$e_n = u_n - u(t_n)$$

then satisfies the error equation

$$\frac{1}{\tau}\sum_{j=0}^{k}\delta_j M(t_{n-j})e_{n-j} + A(t_n)e_n = d_n, \qquad n \geq k.$$

Time-dependent norms and semi-norms

We work with the norm, for $w = (w_i)$ and $w_h(t) = \sum_i w_i \phi_i(\cdot, t)$,

$$|w|_t^2 = w^T M(t) w = \int_{\Gamma_h(t)} w_h(t)^2$$

and the semi-norm

$$\|w\|_t^2 = w^T A(t) w = \int_{\Gamma_h(t)} |\nabla_{\Gamma_h(t)} w_h(t)|^2$$

Basic inequalities:

$$\begin{aligned} w^T \big(M(s) - M(t)\big) z &\leq \mu \, |s - t| \, |w|_t \, |z|_t \\ w^T \big(A(s) - A(t)\big) z &\leq \kappa \, |s - t| \, \|w\|_t \, \|z\|_t \end{aligned}$$

for all $w, z \in \mathbb{R}^N$ and $0 \leq s, t \leq T$.

Stability

For the k-step BDF method with $k \leq 5$,

$$|e_n|_{t_n}^2 + \tau \sum_{j=k}^{n} \|e_j\|_{t_j}^2 \leq C\,\tau \sum_{j=k}^{n} \|d_j\|_{*,t_j}^2 + C \max_{0 \leq i \leq k-1} |e_i|_{t_i}^2$$

where $\|w\|_{*,t}^2 = w^T (A(t) + M(t))^{-1} w$.

C depends on μ, κ, T, but is independent of the spatial grid size h.

Stability: sketch of proof

Rewrite the error equation as

$$M_n \sum_{j=0}^{k} \delta_j e_{n-j} + \tau A_n e_n = \tau d_n + \sum_{j=1}^{k} \delta_j \left(M_n - M_{n-j}\right) e_{n-j}$$

and take the inner product with $e_n - \eta e_{n-1}$.

Use

- the results by Dahlquist and Nevanlinna & Odeh,
- the norm inequalities,
- standard estimates like Cauchy-Schwarz and Young inequality.

Error bound

Let $P_h(t)$ be the Ritz projection to the ESFEM space at time t.

The error $e_h^n = u_h^n - P_h(t_n)u(t_n)$ is bounded for $t_n \leq T$ by

$$\max_{k \leq j \leq n} \|e_h^j\|_{L_2(\Gamma_h(t_j))} + \Big(\tau \sum_{j=k}^{n} \|\nabla_{\Gamma_h(t_j)} e_h^j\|^2_{L_2(\Gamma_h(t_j))}\Big)^{1/2}$$

$$\leq C\tau^k + Ch^2.$$

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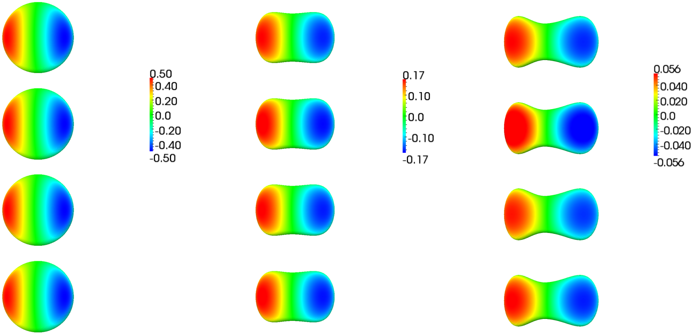
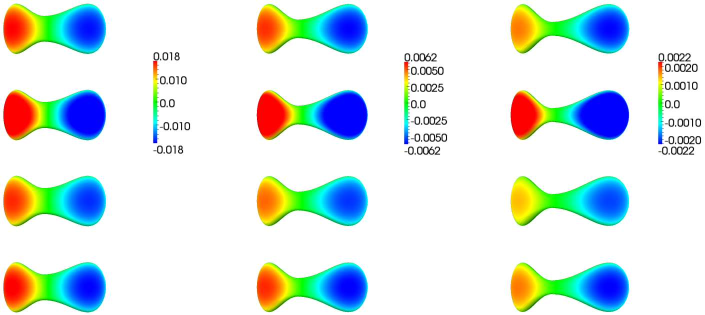


(a) $t = 0$ (b) $t = 0.2$ (c) $t = 0.4$

Reference solution and BDF of orders 1,2,4 with $\tau = 0.05$

Numerical experiment



(d) $t = 0.6$ (e) $t = 0.8$ (f) $t = 1$

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