Operator splitting for delay equations

Bálint Farkas
joint with
András Bátkai, Petra Csomós

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Delay equations

$$\begin{cases} \dot{u}(t) = Bu(t) + g(u(t-1)) \\ u(0,s) = f(s), \quad s \in [-1,0] \quad f : [-1,0] \to H \text{ given initial function} \end{cases}$$

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History function:

$$u_t : [-1,0] \to H,\ u_t(\sigma) := \begin{cases} u(t+\sigma) & t+\sigma \geq 0 \\ f(t+\sigma) \end{cases}$$

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$$\begin{cases} \dot u(t) = Bu(t) + g(\Phi u_t) \\ u_0 = f \end{cases}$$

- $\Phi v = \delta_{-1} v$ point delay

or more generally

- $\Phi v = \int_{-1}^{0} v(\sigma)\,\mathrm{d}\eta(\sigma)$ with $\eta \in \mathrm{BV}([-1,0]; \mathscr{L}(H))$

Cauchy problem:

$$\begin{cases} \dot{u}(t) = Bu(t) + g(\Phi u_t) \\ u_0 = f \end{cases}$$

Sequential splitting

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Split into sub-problems: $\dot{v}(t) = \Psi v_t$, $\Psi = g \circ \Phi$ and $\dot{w}(t) = Bw(t)$

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time step: $h = \frac{t}{n}$

1st step $\begin{cases} \text{Solve problem } \dot{v}(t) = \Psi v_t \text{ with } v_0 = f \\ \text{Solve problem } \dot{w}(t) = Bw(t) \text{ with } w(0) = v(h) \end{cases}$

2nd step $\begin{cases} \text{Solve problem } \dot{v}(t) = \Psi v_t \text{ with } v_h = w_h \\ \text{Solve problem } \dot{w}(t) = Bw(t) \text{ with } w(h) = v(2h) \end{cases}$

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⋮

Iterate this procedure for n steps

⋮

nth step $\begin{cases} \text{Solve problem } \dot{v}(t) = \Psi v_t \text{ with } v_{(n-1)h} = w_{(n-1)h} \\ \text{Solve problem } \dot{w}(t) = Bw(t) \text{ with } w((n-1)h) = v(nh) \end{cases}$

$\rightsquigarrow u^{\text{sp}}(t)$ the splitting-solution at time-level t

G. Webb, A. Bátkai–S. Piazzera:

- $\mathcal{E} := H \times \mathrm{L}^p\big([-1,0];H\big)$ new state space
- and the new unknown function as

$$t \mapsto \mathcal{U}(t) := \begin{pmatrix} u(t) \\ u_t \end{pmatrix} \in \mathcal{E}.$$

Then the delay equation takes the form of an abstract Cauchy problem on the space $\mathcal{E}$

$$\begin{cases} \dot{\mathcal{U}}(t) & = \mathcal{G}\mathcal{U}(t), \qquad t \geq 0, \\ \mathcal{U}(0) & = \binom{x}{f} \in \mathcal{E}, \end{cases}$$

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$$\mathcal{G} := \begin{pmatrix} B & \Psi \\ 0 & \frac{\mathrm{d}}{\mathrm{d}\sigma} \end{pmatrix} \quad \text{with} \quad D(\mathcal{G}) := \left\{ \binom{x}{f} \in D(B) \times \mathrm{W}^{1,p}\big([-1,0];H\big) : \ f(0) = x \right\}$$

- generates a semigroup (dynamical system) $\mathcal{S}$

$\mathcal{S}(t)\mathcal{S}(s) = \mathcal{S}(t+s)$, $\mathcal{S}(0) = I$

- the solution is $\mathcal{U}(t) = \mathcal{S}(t)\binom{x}{f}$ for $\binom{x}{f} \in D(\mathcal{G})$

Splitting for the delay equation

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Splitting Type 1

$$\mathcal{G} := \begin{pmatrix} B & \Psi \\ 0 & \frac{\mathrm{d}}{\mathrm{d}\sigma} \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \Psi \\ 0 & \frac{\mathrm{d}}{\mathrm{d}\sigma} \end{pmatrix} = \mathcal{A}_0 + \mathcal{A}_d$$

Splitting for the delay equation

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Splitting Type 2

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Splitting for the delay equation — Type 1

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Then:

- $\mathcal{G}$, $\mathcal{A}_d$, $\mathcal{A}_0$ are generators of quasi-contraction semigroups, $\mathcal{U}$, $\mathcal{S}_d$, $\mathcal{S}_0$

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Then:

- $\mathcal{G}$, $\mathcal{A}_d$, $\mathcal{A}_0$ are generators of quasi-contraction semigroups, $\mathcal{U}$, $\mathcal{S}_d$, $\mathcal{S}_0$
- The sequential splitting converges:

$$\mathcal{U}(t)\binom{x}{f} = \lim_{n\to\infty} \left[\mathcal{S}_0\left(\tfrac{t}{n}\right)\mathcal{S}_d\left(\tfrac{t}{n}\right)\right]^n \binom{x}{f}.$$

Theorem: [E. Hansen, A. Ostermann]

- H Hilbert space
- A, B generate contraction C_0-semigroups T_0, T_1 on H
- $G = \overline{A_0 + A_1}$ generates a semigroup T and satisfies

$$D(G^2) \subset D(A_0 A_1)$$

$\Longrightarrow$ $$\|T_0(h)T_1(h)x - T(h)x\| \leq C \cdot h^2 \cdot \|x\|_{G^2} \qquad \text{for all } x \in D(G^2)$$

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Application to delay equations

Suppose:

- B generates a contraction semigroup, $g(x) = x$
- $\operatorname{ran} \Phi \subset D(B)$

$$\Longrightarrow \qquad \|\mathcal{S}_0(h)\mathcal{S}_d(h)x - \mathcal{S}(h)x\| \leq C \cdot h^2 \cdot \left\| \binom{x}{f} \right\|_{\mathcal{G}^2} \qquad \text{for all } \binom{x}{f} \in D(\mathcal{G}^2)$$

Suppose:

- $\operatorname{ran}\Phi \subset D(B)$

Need: $D(\mathcal{G}^2) \subseteq D(\mathcal{A}_0\mathcal{A}_d)$

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$$= \left\{ \binom{x}{f} \in D(B) \times \mathrm{W}^{2,p}([-1,0];H) \ : \ f(0) = x, \ f'(0) = Bx + \Phi f \in D(B) \right\}.$$

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$$\begin{aligned} D(\mathcal{G}^2) &= \left\{ \tbinom{x}{f} \in D(\mathcal{G}) \ : \ \mathcal{G}\tbinom{x}{f} \in D(\mathcal{G}) \right\} = \left\{ \tbinom{x}{f} \in D(\mathcal{G}) \ : \ \tbinom{Bx+\Phi f}{f'} \in D(\mathcal{G}) \right\} \\ &= \left\{ \tbinom{x}{f} \in D(B) \times \mathrm{W}^{2,p}([-1,0];H) \ : \ f(0) = x, \ f'(0) = Bx + \Phi f \in D(B) \right\}. \end{aligned}$$

$\Longrightarrow$ $$D(\mathcal{G}^2) \subset D(\mathcal{G}) \subset D(\mathcal{A}_d) = D(\mathcal{A}_0\mathcal{A}_d), \qquad \checkmark$$

- $\Phi(v) = \delta_{-1} v$ (point delay) $\Phi(v) = \int_{-1}^{0} v(\sigma)\,\mathrm{d}\eta(\sigma)$
- $\Phi : \mathrm{W}^{1,p}([1,0];H) \to H$

Drawbacks:

1. $\operatorname{ran}\Phi \subseteq D(B)$ is too strong!!!! point delays are excluded!

Error estimates from abstract results (linear) — Type 1

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- $\Phi : \mathrm{W}^{1,p}([1,0];H) \to H$

Drawbacks:

1. $\operatorname{ran}\Phi \subseteq D(B)$ is too strong!!!! point delays are excluded!

2. Too strong and unnatural compatibility conditions are needed:
$$D(\mathcal{G}^2) = \left\{ \binom{x}{f} \in D(B) \times \mathrm{W}^{2,p}([-1,0];H) \;:\; f(0) = x,\; f'(0) = Bx + \Phi f \in D(B) \right\}.$$

$\binom{x}{f} \in D(\mathcal{G}^2)$ means:

- $f(0) = x$
- AND $Bx + \Phi f = f'(0)$????

Theorem: [A. Bátkai, P. Csomós, B.F.]

Suppose:

- $p \geq 1$, B generates a linear contraction semigroup in the Hilbert space H
- $\eta : [-1,0] \to \mathscr{L}(H)$ Lipschitz near 0
- Φ maps $D(B)$ valued function into $D(B)$
- g is Lipschitz in H and $D(B)$

Let

$$\binom{x}{f} \in \mathcal{D} := \left\{ \binom{x}{f} \in D(B^2) \times \mathrm{W}^{1,p}([-1,0]; D(B)),\ f \in \mathrm{Lip}([-1,0]; H),\ f(0) = x \right\}.$$

$\Longrightarrow$

For all $t_{\max} > 0$ there is a constant $C > 0$ such that

$$\left\| \left(\left(\mathcal{S}_0(\tfrac{t}{n}) \mathcal{S}_d(\tfrac{t}{n}) \right)^n - \mathcal{S}(t) \right) \binom{x}{f} \right\| \leq \frac{C}{n} \left\| \binom{x}{f} \right\|_{\mathcal{D}} \quad \text{for all } t \in [0, t_{\max}],$$

for all $\binom{x}{f} \in \mathcal{D}$. "First order convergence"

Where:

$$\left\| \binom{x}{f} \right\|_{\mathcal{D}} := \|x\|_B + \|Bx\|_B + \|f\|_{\mathrm{W}^{1,p}(D(B))} + \|f\|_{\mathrm{Lip}(H)}.$$

- $\mathcal{D}$ is invariant under $\mathcal{S}$
- Local error analysis for $\binom{x}{f} \in \mathcal{D}$
- Standard telescopic argument
  "stability + local error estimates $\Longrightarrow$ global error estimates"

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Telescopic summation:

$$\begin{aligned}&\big(\mathcal{S}_0(\tfrac{t}{n})\mathcal{S}_d(\tfrac{t}{n})\big)^n - \mathcal{S}(t)\\ &\quad= \sum_{k=0}^{n-1}\Big(\big(\mathcal{S}_0(\tfrac{t}{n})\mathcal{S}_d(\tfrac{t}{n})\big)^{n-k}\underbrace{\mathcal{S}_0(\tfrac{t}{n})\mathcal{S}_d(\tfrac{t}{n})\mathcal{S}(\tfrac{kt}{n}) - \big(\mathcal{S}_0(\tfrac{t}{n})\mathcal{S}_d(\tfrac{t}{n})\big)^{n-k}\mathcal{S}(\tfrac{t}{n})\mathcal{S}(\tfrac{kt}{n})}_{\text{gives the local error}}\Big).\end{aligned}$$

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$$\Big\|\Big(\big(\mathcal{S}_0(\tfrac{t}{n})\mathcal{S}_d(\tfrac{t}{n})\big)^n - \mathcal{S}(t)\Big)\binom{x}{f}\Big\|_{\mathcal{E}} \le \sum_{k=0}^{n-1} K\left(\tfrac{t}{n}\right)^2\Big(1+\big\|\mathcal{S}(\tfrac{kt}{n})\tbinom{x}{f}\big\|_{\mathcal{D}}\Big) \le \frac{C}{n}\Big(1+\big\|\mathcal{S}(\tfrac{kt}{n})\tbinom{x}{f}\big\|_{\mathcal{D}}\Big)$$

Invariance of

$$\mathcal{D} := \left\{ \binom{x}{f} \in D(B^2) \times \mathrm{W}^{1,p}([-1,0]; D(B)),\ f \in \mathrm{Lip}([-1,0]; H),\ f(0) = x \right\}.$$

under the unsplit problem $\mathcal{S}$.

For the proof:

- Lipschitz continuity for solutions of nonlinear dissipative problems
- Lipschitz continuity of η (the delay term)

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Local error

There is a constant $K > 0$ such that for all $h \geq 0$ we have

$$\left\| \mathcal{S}_0(t)\mathcal{S}_d(h)\binom{x}{f} - \mathcal{S}(h)\binom{x}{f} \right\|_{\mathcal{E}} \leq Kh^2 \left\| \binom{x}{f} \right\|_{\mathcal{D}} \quad \text{for all } \binom{x}{f} \in \mathcal{D}.$$

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For the proof:

- structure of $\mathcal{S}_d$ and $\mathcal{S}_0$
- variation of constants formula

Nonautonomous delay equation:

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Nonautonomous delay equation:

$$\begin{cases} \dot{u}(t) = B(t)u(t) + \Phi(t)u_t \\ u_0 = f \end{cases}$$

- $B(s) \in \mathscr{L}(H)$ and generates $V^{(s)}$ on X.
- $\Phi(s):\ \mathrm{L}^1([-1,0];X) \to X$ are bounded for all $s \in \mathbb{R}$.
- $s \mapsto B(s)x$ is bounded and locally Lipschitz continuous, i.e., for all $T_0 > 0$ there is $L_{T_0} \geq 0$ with
$$\|B(s)x - B(t)x\| \leq L_{T_0}\|x\||t-s|$$
for all $|t|, |s| \leq T_0$.
- $s \mapsto \Phi(s)f$ is bounded and locally Lipschitz continuous, i.e., for all $T_0 > 0$ there is $L_{T_0} \geq 0$ with
$$\|\Phi(s)f - \Phi(t)f\| \leq L_{T_0}\|f\|_1|t-s|$$
for all $|t|, |s| \leq T_0$.

Splitting for delay equations — Type 2 (linear)

$$\mathcal{G}(t) := \begin{pmatrix} B(t) & \Phi(t) \\ 0 & \frac{\mathrm{d}}{\mathrm{d}\sigma} \end{pmatrix} = \begin{pmatrix} B(t) & 0 \\ 0 & \frac{\mathrm{d}}{\mathrm{d}\sigma} \end{pmatrix} + \begin{pmatrix} 0 & \Psi(t) \\ 0 & 0 \end{pmatrix} = \mathcal{B}_d(t) + \mathcal{B}_0(t)$$

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Then:

the following problems are well-posed:

- $\dot{\mathcal{V}}(t) = \mathcal{B}_0(s)\mathcal{V}(t)$, $t \geq 0$ $\rightsquigarrow$ semigroup $\mathcal{T}_0^{(s)}$
- $\dot{\mathcal{W}}(t) = \mathcal{B}_d(s)\mathcal{W}(t)$, $t \geq 0$ $\rightsquigarrow$ semigroup $\mathcal{T}_d^{(s)}$
- $\dot{\mathcal{U}}(t) = \mathcal{G}(t)\mathcal{U}(t)$, $t \geq s$ $\rightsquigarrow$ evolution family $\mathcal{T}$

$\mathcal{T}(t,s) = \mathcal{T}(t,r)\mathcal{T}(r,s)$, $\mathcal{T}(s,s) = I$ for all $s \leq r \leq t$

Convergence rate from abstract results — Type 2

Suppose the equations are autonomous

$$\mathcal{G} := \begin{pmatrix} B & \Psi \\ 0 & \frac{\mathrm{d}}{\mathrm{d}\sigma} \end{pmatrix} \quad \text{with} \quad D(\mathcal{G}) := \left\{ \binom{x}{f} \in H \times \mathrm{W}^{1,1}\big([-1,0];H\big) : \ f(0) = x \right\}$$

$$\mathcal{B}_d := \begin{pmatrix} 0 & \Psi \\ 0 & 0 \end{pmatrix} \quad \text{with} \quad D(\mathcal{B}_0) := H \times \mathrm{L}^1\big([-1,\ 0];H\big)$$

$$\mathcal{B}_0 := \begin{pmatrix} B & 0 \\ 0 & \frac{\mathrm{d}}{\mathrm{d}\sigma} \end{pmatrix} \quad \text{with} \quad D(\mathcal{B}_d) := \left\{ \binom{x}{f} \in H \times \mathrm{W}^{1,1}\big([-1,0];H\big) : \ f(0) = x \right\}$$

Suppose the equations are autonomous

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Commutator bounds [cf. T. Jahnke, Ch. Lubich]???

$$\begin{aligned} D(\mathcal{B}_0\mathcal{B}_d) &= \left\{ \tbinom{x}{f} : \mathcal{B}_d \tbinom{x}{f} \in D(\mathcal{B}_0) \right\} = \left\{ \tbinom{x}{f} : \tbinom{\Phi f}{0} \in D(\mathcal{B}_0) \right\} \\ &= \left\{ \tbinom{x}{f} : \Phi f = 0 \right\} \\ D(\mathcal{B}_d\mathcal{B}_0) &= D(\mathcal{B}_0) \end{aligned}$$

$\Longrightarrow$ very bad commutator properties!!!

abstract results are not applicable

Convergence rate for the sequential splitting — Type 2

Theorem: [A. Bátkai, P. Csomós, B. F.]

For every $T_0 > 0$ there is constant $C > 0$ such that for all $f \in \mathrm{W}^{1,1}([-1,0];X)$ and $x = f(0)$ the inequality

$$\Big\|\prod_{j=0}^{n-1}\mathcal{T}_0^{(s+jh)}(h)\mathcal{T}_d^{(s+jh)}(h)\tbinom{x}{f} - \mathcal{T}(t,s)\tbinom{x}{f}\Big\| \leq \frac{C(t-s)^2}{n}\big(\|x\| + \|f\|_1 + \|f'\|_1\big)$$

holds for all $s \in [-T_0, T_0]$, $t \in [s, s+T_0]$ and for all $n \in \mathbb{N}$, where $h = \frac{t-s}{n}$.

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holds for all $s \in [-T_0, T_0]$, $t \in [s, s+T_0]$ and for all $n \in \mathbb{N}$, where $h = \frac{t-s}{n}$.

For the proof:

- stability + telescoping argument +
- **Local error estimate:**

  For $T_0 > 0$ there is a constant $C > 0$ such that

  $$\Big\|\mathcal{T}_d^{(s)}(h)\mathcal{T}_0^{(s)}(h)\binom{x}{f} - \mathcal{U}(s+h,s)\binom{x}{f}\Big\| \le Ch^2\big(\|x\| + \|f\|_1 + \|f'\|_1\big)$$

  holds for all $h \in [0,1]$, $s \in [-T_0, T_0]$ and $f \in \mathrm{W}^{1,1}([-1,0];X)$, $x = f(0)$.

Special structure:

$$\mathcal{T}_0^{(r)}(t) := \begin{pmatrix} V^{(r)}(t) & 0 \\ V_t^{(r)} & L(t) \end{pmatrix} \quad \text{and} \quad \mathcal{T}^{(r)}(t) = \begin{pmatrix} I & t\Phi(r) \\ 0 & I \end{pmatrix},$$

WHERE:

L is the left-shift semigroup on $\mathrm{L}^1([-1,0];H)$

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$$(V_t^{(r)}x)(\sigma) := \begin{cases} V^{(r)}(t+\sigma)x, & \text{if} \quad \sigma \in [-t,0], \\ 0, & \text{if} \quad \sigma \in [-1,-t) \end{cases}$$

Local error for splitting — Type 2

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- The local error can be directly estimated

Delay equation:

$$\begin{cases} \dot{u}(t) = Bu(t) + g(\Phi u_t) \\ u_0 = f \end{cases}$$

Summary

Delay equation:

$$\begin{cases} \dot{u}(t) = Bu(t) + g(\Phi u_t) \\ u_0 = f \end{cases}$$

Splitting Type 1

$$\mathcal{G} := \begin{pmatrix} B & \Psi \\ 0 & \frac{\mathrm{d}}{\mathrm{d}\sigma} \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \Psi \\ 0 & \frac{\mathrm{d}}{\mathrm{d}\sigma} \end{pmatrix} = \mathcal{A}_0 + \mathcal{A}_d$$

Splitting Type 2

$$\mathcal{G}(t) := \begin{pmatrix} B(t) & \Phi(t) \\ 0 & \frac{\mathrm{d}}{\mathrm{d}\sigma} \end{pmatrix} = \begin{pmatrix} B(t) & 0 \\ 0 & \frac{\mathrm{d}}{\mathrm{d}\sigma} \end{pmatrix} + \begin{pmatrix} 0 & \Phi(t) \\ 0 & 0 \end{pmatrix} = \mathcal{B}_0 + \mathcal{B}_d$$

result in a **first order** splitting

Thanks for listening!