Convergence analysis of Strang splitting for Vlasov-type equations

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Motivation

Equations

1. Vlasov equation

$$\partial_t f(t,\mathbf{x},\mathbf{v}) + \mathbf{v}\cdot\nabla f(t,\mathbf{x},\mathbf{v}) + \mathbf{F}\cdot\nabla_{\mathbf{v}} f(t,\mathbf{x},\mathbf{v}) = 0,$$

   with force term $\mathbf{F}$ and particle-density f.

2. Coupling to the EM field (Maxwell or Poisson equations)
3. Approximations (e.g. gyrokinetic equations, linear Vlasov equation)

Applications

1. Plasma simulations
   (e.g. Tokamaks or plasma-laser interactions)
2. Especially if fluid models are not sufficient

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Vlasov-type equations

Definition (Vlasov-type equation)

$$\begin{cases} \partial_t f(t) = (A+B)f(t) \\ f(0) = f_0, \end{cases}$$

where A is a linear operator and $Bf = B(f)f$ with $B(f)$ linear.

Scope

1. Abstract evolution equation
2. Includes Vlasov–Poisson, Vlasov–Maxwell, and gyrokinetic equations as a special case
3. No discretization in space

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Strang splitting

Definition (Strang splitting)

$$S = e^{\frac{h}{2}A} e^{hB_{h/2}} e^{\frac{h}{2}A},$$

where $B_{h/2}$ is a linear approximation of order 1 to Bf.

Strang splitting for grid-based Vlasov solvers

1. Vlasov–Poisson equations (Cheng and Knorr 1976)
2. Vlasov–Maxwell equations (Mangeney et al. 2002)
3. Computationally interesting since for

$$B_{h/2}f = B(f_{h/2})f$$

solution can be represented as a translation.

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Convergence

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1. Convergence follows from consistency and stability

Stability

1. Stability follows from probability conservation

Consistency

1. Expansion of the exact solution (Gröbner–Alekseev formula)
2. Expansion of the splitting operator
3. Estimation of the (possibly) unbounded remainder terms

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Expansion of the exact solution

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\begin{aligned}
f(h) =& E_B(h, f_0) + \int_0^h \partial_2 E_B(h-\tau, f(\tau)) A E_B(\tau, f_0)\,\mathrm{d}\tau \\
&+ \int_0^h \int_0^\tau \partial_2 E_B(h-\tau, f(\tau)) A \partial_2 E_B(\tau-\sigma, f(\sigma)) A E_B(\sigma, f_0)\,\mathrm{d}\sigma \mathrm{d}\tau \\
&+ \int_0^h \int_0^{\tau_1} \int_0^{\tau_2} \left(\prod_{k=0}^{2} \partial_2 E_B(\tau_k - \tau_{k+1}, f(\tau_{k+1})) A \right) f(\tau_3)\,\mathrm{d}\tau_1 \mathrm{d}\tau_2 \mathrm{d}\tau_3,
\end{aligned}
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where $\tau_0 := h$.

Expansion of the splitting operator

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Sf_0 = e^{hB_{h/2}} f_0 + \frac{h}{2} \left\{ A, e^{hB_{h/2}} \right\} f_0 + \frac{h^2}{8} \left\{ A, \left\{ A, e^{hB_{h/2}} \right\} \right\} f_0 + R_3 f_0.
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Bounds

1. Compare the terms by employing a quadrature rules
2. We have to estimate e.g.

$$\left[e^{hB_{h/2}} - \partial_2 E_B(h, f_0)\right] Af_0 = \partial_2 \left[e^{hB_{h/2}} - E_B(h, f_0)\right] Af_0$$

3. Use Gröbner–Alekseev formula to get

$$E_B(h, f_0) - e^{hB_{h/2}} f_0 = \int_0^h e^{(h-\tau)B_{h/2}} (B - B_{h/2}) E_B(\tau, f_0) \, \mathrm{d}\tau$$

4. Gives a condition on $B - B_{h/2}$.
5. Note that f_0 is an arbitrary initial value for the Strang splitting operator

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Consistency

Theorem (Consistency)

Suppose that $B_{h/2}$ is an approximation of order 1 to B and the estimates

$$\|A^i e^{(h-s)B_{h/2}}(R_{2-i}(B) - R_{2-i}(B_{h/2}))E_B(s, f_0)\|_X \leq C, \quad i \in \{0,1,2\} \tag{1}$$

$$\|(R_1(B_{h/2}) - R_1(\partial_2 B f_0))A f_0\|_X \leq C, \tag{2}$$

$$\|A^{\delta_{i2}} B_{h/2}^{1+\delta_{i0}} \varphi_{1+\delta_{i0}}(hB_{h/2}) A^{1+\delta_{i1}} f_0\|_X \leq C, \quad i \in \{0,1,2\} \tag{3}$$

$$\|A^{\delta_{i2}} R_{1+\delta_{i0}}(\partial_2 E_B(\cdot, f_0)) A^{1+\delta_{i1}} f_0\|_X \leq C, \quad i \in \{0,1,2\} \tag{4}$$

hold uniformly in t and in $s \in [0,h]$. In addition, suppose that for all $k_j \in \mathbb{N}$, with $\sum_{j=1}^{i+1} k_j = 3 - i$, the estimates

$$\left\| \left(\prod_{j=1}^{i} D_j^{k_j} \partial_2 E_B(s_j, f(\sigma_j))A \right) \partial_{s_{i+1}}^{k_{i+1}} E_B(s_{i+1}, f_0) \right\|_X \leq C, \quad i \in \{1,2\} \tag{5}$$

$$\left\| \left(\prod_{k=1}^{3} \partial_2 E_B(s_k - \sigma_k, f(\sigma_k))A \right) f(s) \right\|_X \leq C, \tag{6}$$

$$\left\| \left\{ A, \left\{ A, \left\{ A, e^{\frac{s}{2}A} e^{hB_{h/2}} e^{\frac{s}{2}A} \right\} \right\} \right\} f_0 \right\|_X \leq C, \tag{7}$$

hold uniformly in t as well as in $s \in [0,h]$, $s_j \in [0,h]$, and $\sigma_j \in [0,h]$, where $D_j^{k_j}$ is a differential operator of order k_j in the variables s_j and σ_j. **Then Strang splitting is consistent of order 2.**

Vlasov–Poisson equations

Definition (Vlasov–Poisson equations in 1+1 dimensions)

$$\begin{cases} \partial_t f(t,x,v) = -v\partial_x f(t,x,v) - E(f(t,\cdot,\cdot),x)\partial_v f(t,x,v) \\ \partial_x E(f(t,\cdot,\cdot),x) = \int_{\mathbb{R}} f(t,x,v)\,\mathrm{d}v - 1 \\ f(0,x,v) = f_0(x,v), \end{cases}$$

Theorem (Uniqueness, existence, and regularity)

Assume that $f_0 \in \mathcal{C}^m_{\mathrm{per,c}}$ is non-negative, then $f \in \mathcal{C}^m(0,T;\mathcal{C}^m_{\mathrm{per,c}})$ and $E(f(t,\cdot,\cdot),x)$ as a function of (t,x) lies in $\mathcal{C}^m(0,T;\mathcal{C}^m_{\mathrm{per}})$. In addition, we can find a $Q(T)$ such that for all $t \in [0,T]$ and $x \in \mathbb{R}$ it holds that $\operatorname{supp} f(t,x,\cdot) \subset [-Q(T),Q(T)]$.

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Derivative with respect to the initial value

Problem

1. To bound

$$\int_0^h \int_0^{\tau_1} \int_0^{\tau_2} \left(\prod_{k=0}^{2} \partial_2 E_B(\tau_k - \tau_{k+1}, f(\tau_{k+1}))A \right) f(\tau_3)\, \mathrm{d}\tau_1 \mathrm{d}\tau_2 \mathrm{d}\tau_3,$$

investigate $\partial_2 E_B(t, u_0)g$ as a function of u_0 and g

2. Methods of characteristics expresses solution in the form

$$V'_{u_0}(t) = E(u(t,\cdot,\cdot),x)$$
$$u(t,x,v) = u_0(x, V_{u_0}(t)(x,v)).$$

Theorem

The following function is well defined

$$\mathcal{C}^m_{\mathrm{per,c}} \times \mathcal{C}^n_{\mathrm{per,c}} \to \mathcal{C}^{\min(m-1,n)}_{\mathrm{per,c}}$$
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Convergence

Consistency

1. Application of A, B, $B_{h/2}$ are maps from $\mathcal{C}^m_{\text{per,c}}$ to $\mathcal{C}^{m-1}_{\text{per,c}}$
2. Control derivatives of $Bf(t)$ with respect to time
3. Control the derivative $\partial_2 E_B(t, u_0)g$

Stability

Rewrite splitting step, for example, as

$$e^{-hE(f_{h/2},x)\partial_v} f_0(x,v) = f_0\left(x, v - E(f_{h/2},x)h\right).$$

Theorem (Convergence)

Suppose $f_0 \in \mathcal{C}^3_{\text{per,c}}$, f_0 is non-negative and $f_{h/2}$ is an approximation to $f(h/2)$ of order 1. Then Strang splitting for the Vlasov–Poisson equations is convergent of order 2 (with respect to the L^1 norm).

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Landau damping

Definition (Landau damping in 1+1 dimensions)

The Vlasov–Poisson equations together with the initial value

$$f_0(x, v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2} (1 + \alpha \cos(0.5x))$$

on the domain $[0, 4\pi] \times \mathbb{R}$.

Landau damping

1. Popular test problem
2. Linear Landau damping ($\alpha = 0.01$)
3. Non-linear Landau damping ($\alpha = 0.5$)

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2. Linear Landau damping ($\alpha = 0.01$)
3. Non-linear Landau damping ($\alpha = 0.5$)

Landau damping

Definition (Landau damping in 1+1 dimensions)

The Vlasov–Poisson equations together with the initial value

$$f_0(x,v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2} (1 + \alpha \cos(0.5x))$$

on the domain $[0, 4\pi] \times \mathbb{R}$.

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Non-linear Landau damping

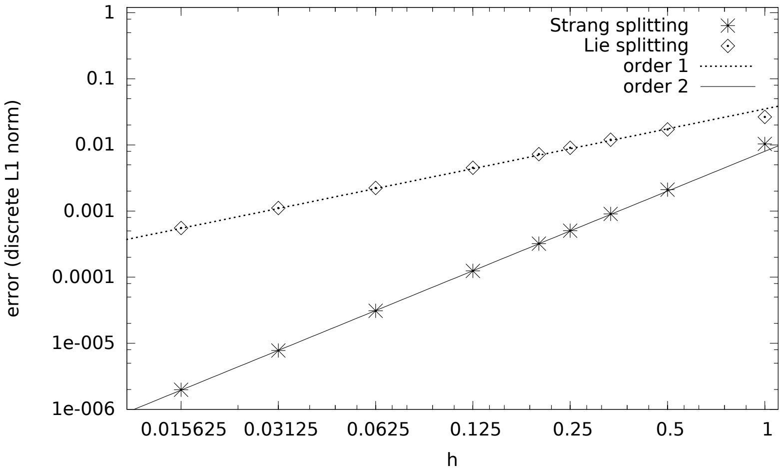
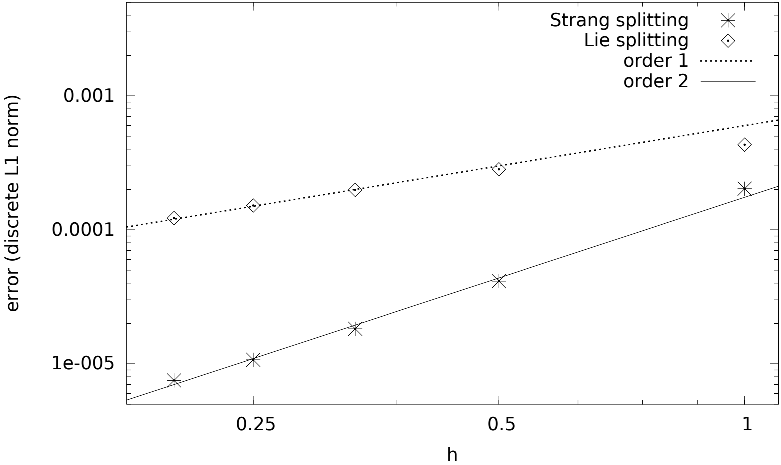


Figure: Approximation compared to a reference solution at $t = 1$. Discontinuous Galerkin approximation in space (order 2, $N_x, N_v = 80$).

Linear Landau damping



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Conclusion

Thank you for your attention