Operator splitting for delay equations, part I

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Outline

1. Abstract delay equations
2. Operator splitting
3. Delay equations as abstract Cauchy problems
4. How to split?
5. Convergence theorems
6. Numerical examples

Abstract delay equations

Idea: The actual state of a system depends on the past as well

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History function:	$w_t(\sigma) := w(t+\sigma), \quad \sigma \in [-1, 0]$

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$\Longrightarrow$	$\frac{\mathrm{d}}{\mathrm{d}t}w = \Phi w_t$
with	$\Phi g = g(-1)$

Abstract delay equations

Idea: The actual state of a system depends on the past as well

Example: $\frac{\partial}{\partial t} w(t,x) = \Delta w(t,x) + w(t-1,x)$

Initial condition: $w(0,x) =$ given

and $w(\sigma,x) =$ given for $\sigma \in [-1,0]$

History function: $w_t(\sigma,x) := w(t+\sigma,x), \quad \sigma \in [-1,0]$

$\Longrightarrow$ $\frac{\partial}{\partial t} w(t,x) = \Phi w_t(x)$

with $\Phi g(x) = g(-1,x)$

Abstract delay equations

In general: X Banach, G operator, $1 \leq p < \infty$:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} u(t) = Gu(t) + \Phi u_t, \quad t \geq 0 \\ u(0) \in X \\ u_0 \in \mathrm{L}^p\big([-1,0], X\big) \end{cases}$$

with $\Phi : \mathrm{L}^p\big([-1,0], X\big) \to X$

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$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = Gu(t) + \Phi u_t$$

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$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = Gu(t) + \Phi u_t$$

It would be easier to solve them separately...

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = Gu(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} v(t) = \Phi v_t$$

Abstract Cauchy problem: X Banach

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---

Sequential spl. $u(t) \approx \left(S(t/n)T(t/n)\right)^n u(0)$

Strang spl. $u(t) \approx \left(T(t/2n)S(t/n)T(t/2n)\right)^n u(0)$

Operator splittings

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} u(t) = A u(t), \quad t \in [0, h] \\ u(0) \quad \text{given} \end{cases}$$

Operator splittings

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u(t) = Au(t), & t \in [0,h] \\ u(0) & \text{given} \end{cases}$$

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}v(t) = Bv(t), & t \in [0,h] \\ v(0) = \boxed{\mathrm{u(h)}} \end{cases}$$

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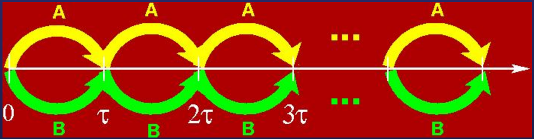
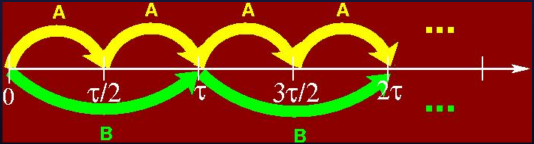
Operator splittings

Sequential



Strang



Theorem

Suppose

$$\begin{aligned} A &\rightsquigarrow T(\cdot) \\ B &\rightsquigarrow S(\cdot) \\ A+B &\rightsquigarrow U(\cdot) \quad C_0\text{-semigroups.} \end{aligned}$$

Then: *splittings convergent* $\Longleftrightarrow$ *stable*

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Then: *splittings convergent* $\Longleftrightarrow$ *stable, i.e.*

there exist $M \geq 1$, $\omega \in \mathbb{R}$:

$$\left\| \left(S(t/n)T(t/n) \right)^n \right\| \leq M e^{\omega t} \quad \text{for all } t \geq 0,\ n \in \mathbb{N}.$$

Delay equations as ACPs

X Banach

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} u(t) = Gu(t) + \Phi u_t, & t \geq 0 \\ u(0) \in X \\ u_0 \in \mathrm{L}^p\big([-1,0], X\big) \end{cases}$$

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let us observe

$$\frac{\partial}{\partial t}u(t+\sigma) = \frac{\partial}{\partial \sigma}u(t+\sigma)$$

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$$\frac{\partial}{\partial t} u_t(\sigma) = \frac{\partial}{\partial \sigma} u_t(\sigma)$$

transport equation

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Let $\mathcal{E} = X \times \mathrm{L}^p\big([-1,0],X\big)$

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$$\frac{\partial}{\partial t}u_t(\sigma) = \frac{\partial}{\partial \sigma}u_t(\sigma)$$

$$\mathcal{U}(t) = \begin{pmatrix} u(t) \\ u_t \end{pmatrix}$$

transport equation

$$\mathcal{G} = \begin{pmatrix} G & \Phi \\ 0 & \frac{\mathrm{d}}{\mathrm{d}\sigma} \end{pmatrix}$$

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Let $\mathcal{E} = X \times \mathrm{L}^p\big([-1,0],X\big)$

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{U}(t) = \mathcal{G}\mathcal{U}(t), & t \geq 0 \\ \mathcal{U}(0) = \binom{u(0)}{u_0} \end{cases}$$

with

$$\mathcal{U}(t) = \begin{pmatrix} u(t) \\ u_t \end{pmatrix}$$

$$\mathcal{G} = \begin{pmatrix} G & \Phi \\ 0 & \frac{\mathrm{d}}{\mathrm{d}\sigma} \end{pmatrix}$$

How to split?

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$$\mathcal{G} = \mathcal{A} + \mathcal{B}$$

$$\begin{pmatrix} G & \Phi \\ 0 & \frac{d}{d\sigma} \end{pmatrix} = \begin{pmatrix} G & 0 \\ 0 & \frac{d}{d\sigma} \end{pmatrix} + \begin{pmatrix} 0 & \Phi \\ 0 & 0 \end{pmatrix}$$

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Convergence theorems – Case [Φ]

$$\begin{pmatrix} G & \Phi \\ 0 & \frac{\mathrm{d}}{\mathrm{d}\sigma} \end{pmatrix} = \begin{pmatrix} G & 0 \\ 0 & \frac{\mathrm{d}}{\mathrm{d}\sigma} \end{pmatrix} + \begin{pmatrix} 0 & \Phi \\ 0 & 0 \end{pmatrix}$$

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$$\downarrow \qquad\qquad \downarrow$$

$$\begin{pmatrix} T(t) & 0 \\ T_t & L(t) \end{pmatrix} \qquad \begin{pmatrix} I & t\Phi \\ 0 & I \end{pmatrix}$$

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with Φ bounded

$G \rightsquigarrow T(\cdot)$

$(T_t f)(\sigma) = T(t+\sigma)$ for $\sigma \in [-t, 0)$, otherwise $= 0$

$L(\cdot)$ left shift semigroup

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$$\mathcal{T}(t) \qquad\qquad \mathcal{S}(t)$$

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$L(\cdot)$ left shift semigroup

Theorem

Suppose $G \rightsquigarrow T(\cdot)$ semigroup, Φ bounded.

Then the splittings are convergent for delay.

Proof.

Reminder: convergence $\Longleftrightarrow$ stability

$$\left\| \left(\mathcal{S}\left(\tfrac{t}{n}\right) \mathcal{T}\left(\tfrac{t}{n}\right) \right)^n \right\| \leq M e^{\omega t}$$

□

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$$\left\| \left(\begin{pmatrix} I & \frac{t}{n}\Phi \\ 0 & I \end{pmatrix} \begin{pmatrix} T(\frac{t}{n}) & 0 \\ T_{\frac{t}{n}} & L(\frac{t}{n}) \end{pmatrix} \right)^n \right\| \le M e^{\omega t}$$

✔

□

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$$\downarrow \qquad\qquad \downarrow$$

$$\mathcal{T}(t) \qquad\qquad \mathcal{S}(t)$$

with some more assumptions

Assumptions

1. $X = H$ Hilbert
2. $G \rightsquigarrow T(\cdot)$ contraction
3. $\Phi : \mathrm{W}^{1,p}\big([-1,0],H\big) \to H$: $\quad \Phi = g \circ \Psi$

   with g Lipschitz continuous

   and $\Psi(f) = \int_{-1}^{0} \mathrm{d}\eta(\sigma) f(\sigma)$ for $f \in \mathrm{C}([-1,0],H)$

   where $\eta : [-1,0] \to \mathcal{L}(H)$ of bounded variation

   and $\eta(-1) = 0$, $\lim\limits_{\sigma \to -1} \eta(\sigma) \neq 0$

Theorem (Brezis-Pazy, 1970, Kobayashi, 1987)

Suppose $\mathcal{A}+\mathcal{B}$ closed,

$\mathcal{A}$, $\mathcal{B}$, $\mathcal{A}+\mathcal{B}$ ω-m-dissipative generators.

Then the sequential splitting is convergent.

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Proposition (Webb, 1976)

$\exists$ new equivalent norm in $\mathcal{E}$: $\mathcal{G}-\gamma\mathcal{I}$ m-dissipative

(γ can be determined)

Corollary

$\mathcal{A}$ and $\mathcal{B}$ "special cases" of $\mathcal{G}$

$\Longrightarrow$ $\exists\gamma$ s.t. $\mathcal{A}$, $\mathcal{B}$, $\mathcal{G}$ γ-m-dissipative

$\Longrightarrow$ sequential splitting convergent ✔

Non-autonomous case:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u(t) = G(t)u(t) + \Phi u_t, \quad t \geq s \\ u(s) \in X \\ \quad u_s \in \mathrm{L}^p\big([-1,0], X\big) \end{cases}$$

Convergence theorems – Case $[\Phi(t)]$

$$\begin{pmatrix} G(r) & \Phi(t) \\ 0 & \frac{\mathrm{d}}{\mathrm{d}\sigma} \end{pmatrix} = \begin{pmatrix} G(r) & 0 \\ 0 & \frac{\mathrm{d}}{\mathrm{d}\sigma} \end{pmatrix} + \begin{pmatrix} 0 & \Phi(r) \\ 0 & 0 \end{pmatrix}$$

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with $u_s \in \mathrm{L}^1\big([-1,0],X\big)$

$\Phi(r)$ bounded for all $r \in \mathbb{R}$

$G(r) \rightsquigarrow T^{(r)}(\cdot)$

$(T_t^{(r)} f)(\sigma) = T^{(r)}(t+\sigma)$ for $\sigma \in [-t,0)$, otherwise $= 0$

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Further assumptions

1. $s \mapsto \Phi(s)f$ bounded and continuous $\forall f \in \mathrm{L}^1\big([-1,0],X\big)$
2. $D(G(s)) =: D$ for all $s \in \mathbb{R}$
3. $s \to R(1,G(s))y$ continuous $\forall y \in X$

Theorem for general non-autonomous problems

The sequential splitting is convergent if it is stable.

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The sequential splitting is convergent if it is stable.

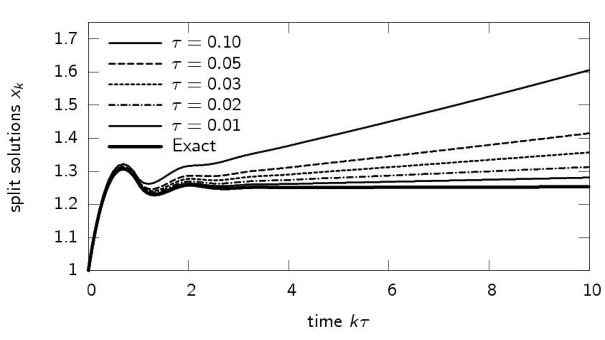
Corollary (delay)

$$\sup_{s\in\mathbb{R}} \left\| \prod_{p=n}^{1} \mathcal{S}^{(s-p\frac{t}{n})}\left(\tfrac{t}{n}\right)\mathcal{T}^{(s-p\frac{t}{n})}\left(\tfrac{t}{n}\right) \right\| \leq Me^{\omega t}$$

(almost) the same computation as in case [Φ]

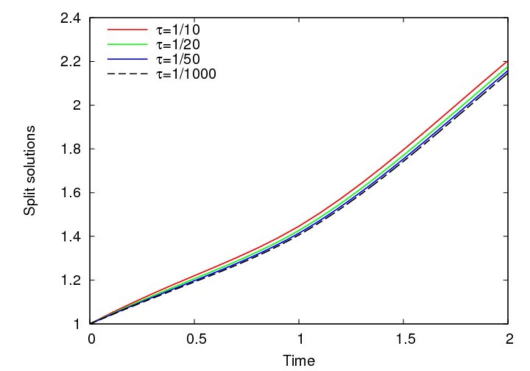
Numerical experiments – case [Φ]

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u(t) = -u(t) + \int_{-1}^{-0.1} u(t+\sigma)\mathrm{d}\sigma, & t \geq 0 \\ u(0) = 1, \quad u_0(\sigma) = 1-\sigma \end{cases}$$



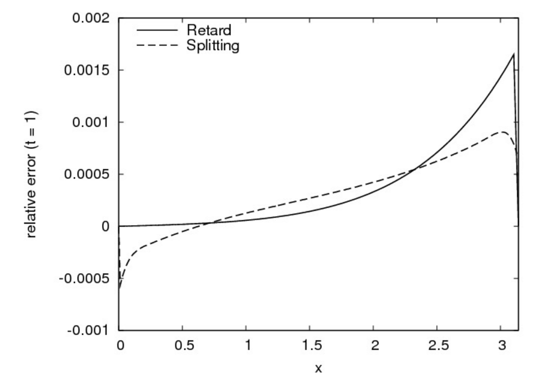
Numerical experiments – case $[\Phi(t)]$

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u(t) = -u(t) + \int_{-1}^{0} (1-\sin t)u(t+\sigma)\mathrm{d}\sigma, & t \geq 0 \\ u(0) = 1, \quad u_0(\sigma) = 1-\sigma \end{cases}$$



Numerical experiments – case [G]

$$\begin{cases} \frac{\partial}{\partial t} w(t,x) = \Delta w(t,x) + w(t-1,x), & t \geq 0,\ x \in [0,\pi] \\ u(0,x) = x(\pi - x), & u_0(\sigma) = (2 + x(\pi - x)) \cdot e^{\sigma+1} \end{cases}$$



Summary

- Delay equations $\longrightarrow$ ACP
- Operator splittings for ACP
- Convergence for delay:
  - bounded Φ
  - unbounded Φ, dissipative case
  - bounded Φ, non-autonomous case
- Order of convergence? See part II $\smile$