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Most unstable trajectories of linear switched systems

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Discrete-time linear switched systems (LSS)

We consider the linear switched system (for $n = 0, 1, \ldots$)

$$x(n+1) \;=\; A_{\sigma(n)}\, x(n), \qquad \sigma : \mathbb{N} \longrightarrow \mathcal{I} := \{1, 2, \ldots, m\}$$

where $x(0) \in \mathbb{R}^k$ and $A_{\sigma(n)} \in \mathbb{R}^{k \times k}$ is an element of the finite (this simplifies presentation) family of matrices

$$\mathcal{F} = \{A_i\}_{i \in \mathcal{I}}$$

associated to the system and σ denotes the switching law.

We are interested in the following issues:

- Stability properties of the solutions in terms of joint spectral radius of the associated family $\mathcal{F}$.
- Geometry of most unstable solutions.

Example in control theory

For a matrix valued function $B : \{1, \ldots, m\} \to \mathbb{R}^{k\times k}$ and a control function $u : (t_0, +\infty) \to \{1, \ldots, m\}$, we consider the linear discontinuous system of ODEs

$$\dot{x}(t) = B(u(t))\, x(t),\ t > t_0, \quad x(t_0) = x_0.$$

We discretize it as follows: given a uniform grid $\{t_n\}$, where $t_{n+1} - t_n = \Delta t$, the discretized control function $u_\Delta(t)$ assumes constant values in each subinterval $(t_n, t_{n+1}]$ of the grid.

Thus the discretized solution $x_\Delta(t)$ satisfies

$$x_\Delta(t_{n+1}) = \mathrm{e}^{\Delta t B(u_\Delta(t_n))} x_\Delta(t_n), \quad x_\Delta(t_0) = x_0,$$

which is of the previous type with $x(n) = x_\Delta(t_n)$ and $A_{\sigma(n)} = \mathrm{e}^{\Delta t B(u_\Delta(t_n))}$, $n = 1, \ldots, m$.

Stability issues: worst case analysis

Aim: determining the most unstable switching law (MUSL), i.e., the law σ giving the solution with highest rate of growth ρ. Specifically we look for a law σ and a norm $\|\cdot\|$ such that

$$\|x(n)\| = \rho^n \|x(0)\| \qquad \text{for all } n.$$

The MUSL can be characterized using optimal control techniques. The variational approach leads to a Hamilton–Jacobi– Bellman equation.

Its solution is referred to as a Barabanov norm of the LSS.

“Although the Barabanov norm was studied extensively, it seems that there are only few examples where it was actually computed in closed form” (**Teichner and Margaliot ’12**).

Worst case analysis: joint spectral radius

In order to analyze all possible solutions, we consider the difference inclusion

$$x(n+1) \in \{A_i\, x(n) \mid i \in \mathcal{I}\}$$

The maximal growth rate of the trajectories associated to the previous difference inclusion turns out to be the so called

joint spectral radius

of the associated family of matrices $\mathcal{F} = \{A_i\}_{i \in \mathcal{I}}$.

If this is less than 1 we have uniform asymptotic stability, i.e.,

$$\lim_{n \to \infty} x(n) = 0 \quad \forall\, x(0)$$

for any possible sequence $\{A_{i_n}\}_{n \geq 1}$ (**Berger & Wang ’92**).

The multiplicative semigroup

We consider the set of products of degree n,

$$\Sigma_n(\mathcal{F}) \;=\; \{A_{i_n} \ldots A_{i_1} \mid i_1, \ldots, i_n \in \mathcal{I}\}$$

and define the product semigroup

$$\Sigma(\mathcal{F}) \;=\; \bigcup_{n \geq 1} \Sigma_n(\mathcal{F}).$$

Generalizing the spectral radius

(1) Joint spectral radius (**Rota & Strang ’60**):

$$\widehat{\rho}(\mathcal{F}) \;=\; \limsup_{n\to\infty} \widehat{\rho}_n(\mathcal{F})^{1/n} \quad \text{with } \widehat{\rho}_n(\mathcal{F}) = \max_{P\in\Sigma_n(\mathcal{F})} \|P\|$$

(2) Generalized spectral radius (**Daubechies** *et al.* **’92**):

$$\overline{\rho}(\mathcal{F}) \;=\; \limsup_{n\to\infty} \overline{\rho}_n(\mathcal{F})^{1/n} \quad \text{with } \overline{\rho}_n(\mathcal{F}) = \max_{P\in\Sigma_n(\mathcal{F})} \rho(P)$$

(3) Common spectral radius (**Elsner ’95**):

$$\nu(\mathcal{F}) \;=\; \inf_{\|\cdot\|\in\mathcal{N}} \|\mathcal{F}\| \quad \text{with} \quad \|\mathcal{F}\| = \max_{i\in\mathcal{I}} \|A_i\|$$

where $\mathcal{N}$ is the set of operator norms.

All these quantities are equal to the same number $\rho(\mathcal{F})$

Framework

Daubechies & Lagarias proved the following inequality, where P is any product of degree d and $\|\cdot\|$ any operator norm:

$$\rho(P)^{1/d} \le \rho(\mathcal{F}) \le \|\mathcal{F}\|$$

Definitions

1. We say that $\mathcal{F}$ has the finiteness property if there exists a spectrum maximizing product, that is a product P for which the left inequality is an equality.
2. We say that $\mathcal{F}$ is nondefective if there exists an operator norm for which the right inequality becomes an equality. Such norm is called an extremal norm.

Extremal Barabanov norms

Definition [Barabanov norm]
We say that an extremal norm $\|\cdot\|$ for the family $\mathcal{F}$ is a Barabanov norm if

$$\max_{i \in \mathcal{I}} \|A_i x\| \; = \; \rho(\mathcal{F}) \|x\| \quad \forall x \in \mathbb{R}^k.$$

Barabanov norms identify - for any initial vector - a most unstable solution associated to a MUSL.

Theorem (**Barabanov ’88**)
Assume that a family of matrices $\mathcal{F}$ is irreducible. Then there exists a Barabanov norm for $\mathcal{F}$.

As a consequence, the existence of a Barabanov norm appears generic as well as the existence of a MUSL.

Computational framework

Recent algorithms proposed in the literature start from the guess of a candidate spectrum maximizing product and attempt to obtain an extremal norm.

Assumptions

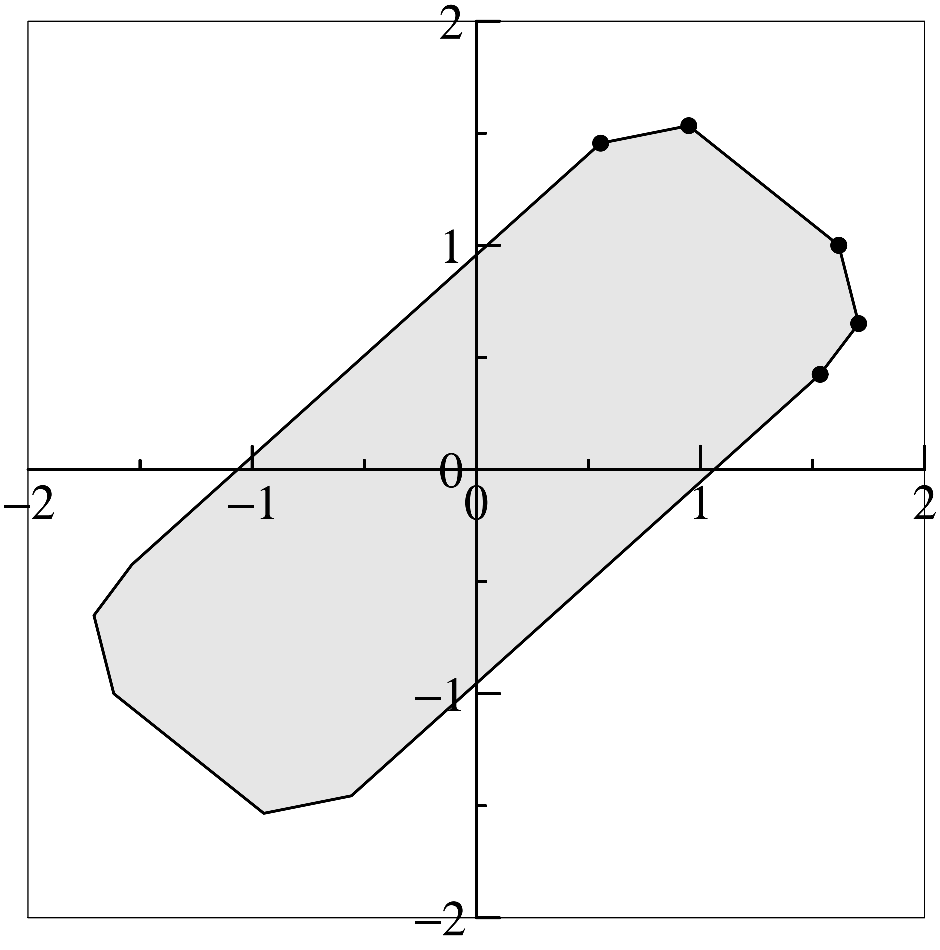
(i) Since the joint spectral radius $\rho(\mathcal{F})$ is a positively homogenous function of the set of matrices, i.e., $\rho(\alpha\mathcal{F}) = |\alpha|\rho(\mathcal{F})$, we assume $\rho(\mathcal{F}) = 1$.

(ii) We assume that $\mathcal{F}$ is nondefective and has the finiteness property.

These imply that there exists P_* such that $\rho(P_*) = 1$ and a norm $\| \cdot \|_*$ such that $\|\mathcal{F}\|_* = 1$.

The polytope algorithms

These algorithms (see, e.g., **Guglielmi, Wirth & Z. ’05** and **Guglielmi & Protasov ’13**) attempt to compute an extremal polytope norm, that is an extremal norm whose unit ball is a centrally symmetric polytope $\mathcal{P}$.



Starting from a suitable initial vector (the leading eigenvector x of the spectrum maximizing product P_*), the algorithm computes $\mathcal{P}$ recursively.

The polytope algorithm

Notation: for a set of vectors $V = \{v_1, \ldots, v_p\}$, $\mathcal{F}\, V$ denotes the set $\{A_i v_j\}_{i,j}$ and $\mathrm{absco}(V) = \mathrm{convhull}\,(\pm v_1, \ldots, \pm v_p)$.

Algorithm 1: Basic algorithm

Data: $\mathcal{F}, x$
Result: $\mathcal{P}$
begin
  Set $V_0 := \{x\}$ and $i = 0$
  while $\mathcal{F}\, V_i \not\subseteq \mathcal{P}^{(i)}$ **do**
    Set $i = i + 1$ and compute $U_i = \mathcal{F}\, V_{i-1}$
    Determine an essential system of vertices V_i of $\mathrm{absco}(V_{i-1} \cup U_i)$
    Set $\mathcal{P}^{(i)} = \mathrm{absco}(V_i)$
  Return $\mathcal{P} := \mathcal{P}^{(i)}$ (extremal polytope unit ball)

Example 1

Let $\mathcal{F} = \{A_1, A_2\}$

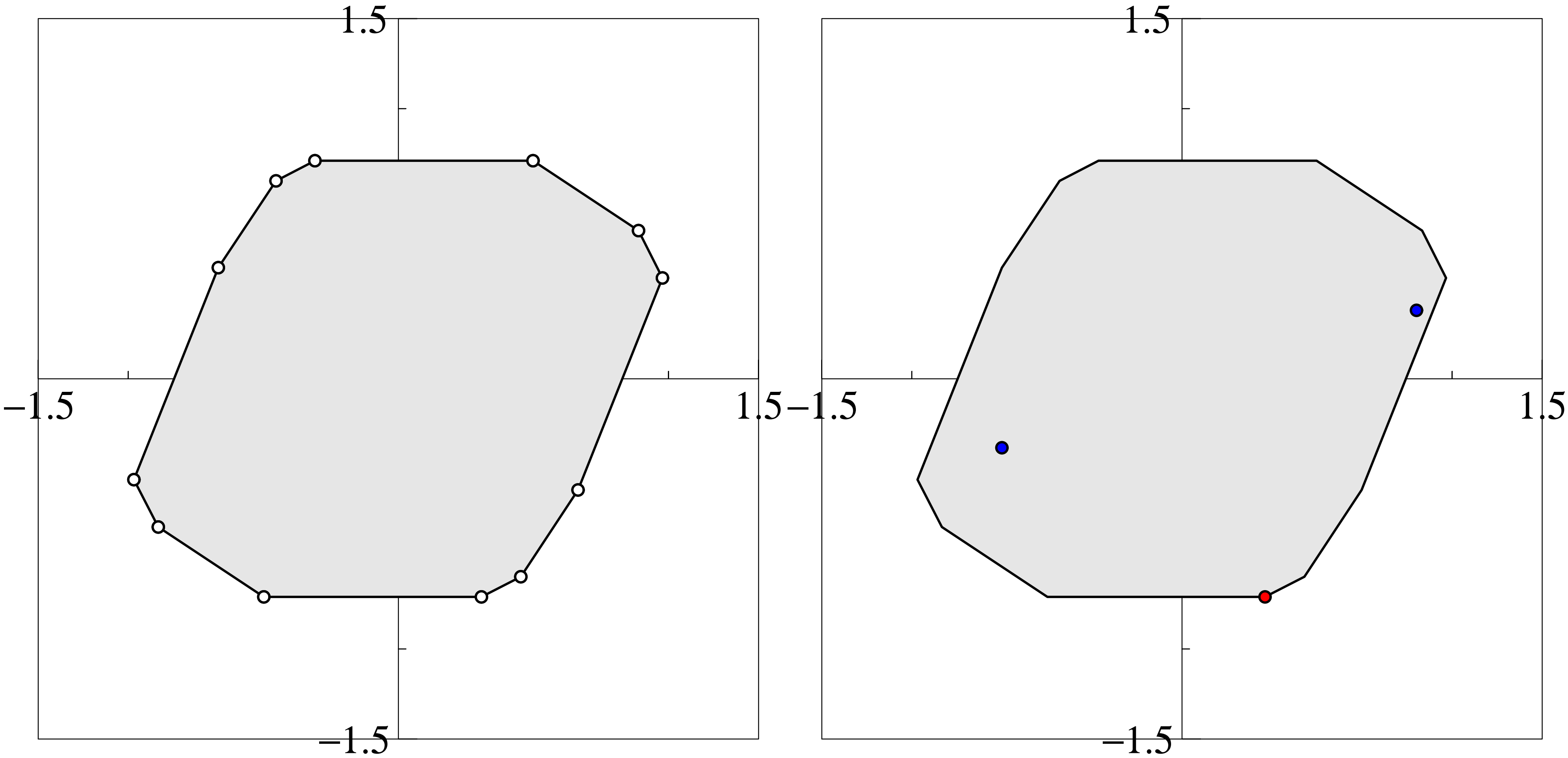
$$A_1 = \alpha \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

with $\alpha = \left(\frac{3+\sqrt{5}}{2}\right)^{-1/5}$, which has spectral radius $\rho(\mathcal{F}) = 1$ and spectrum maximizing product $P_* = A_1 A_2 A_1^2 A_2$.

Applying the polytope algorithm yields an extremal polytope norm after 5 steps, whose unit ball $\mathcal{P}$ is a polytope with 6 vertices.

Is this a Barabanov norm?

Computed extremal polytope norm



In the right picture a boundary point x is drawn in red and the transformed vectors A_1x and A_2x are drawn in blue. Therefore, this is not a Barabanov norm.

Duality

Definition [adjoint polytope]

Let $\mathcal{P}$ be a real centrally symmetric polytope, i.e., there exists a set of vectors $V = \{v_1, \ldots, v_p\}$ such that

$$\mathcal{P} = \operatorname{convhull}\left(\pm v_1, \ldots, \pm v_p\right).$$

Then we call adjoint (or dual) of $\mathcal{P}$ the polytope

$$\mathcal{P}^* = \operatorname{adj}(V) = \left\{ x \in \mathbb{R}^k \;\middle|\; \left|\langle x, v_i \rangle\right| \leq 1,\ i = 1, \ldots, p \right\}.$$

Proposition

Let $\mathcal{P}$ and $\mathcal{P}^*$ be a polytope and its ajoint and let $\|\cdot\|_{\mathcal{P}}$ and $\|\cdot\|_{\mathcal{P}^*}$ be the associated norms. Then, for any matrix A,

$$\|A\|_{\mathcal{P}^*} = \|A^{\mathrm{T}}\|_{\mathcal{P}} \quad (\text{hence } \|\mathcal{F}\|_{\mathcal{P}^*} = \|\mathcal{F}^{\mathrm{T}}\|_{\mathcal{P}}).$$

How to get a Barabanov extremal norm

Main observation: the polytope algorithm determines a polytope $\mathcal{P} = \mathrm{convhull}\,(\pm v_1, \ldots, \pm v_p)$ characterized by

$$v_\ell \;=\; A_{i_\ell} v_{j_\ell} \qquad \text{for some } \; j_\ell \in \{1, \ldots, p\} \;\&\; i_\ell \in \{1, \ldots, m\}.$$

Therefore, with $A_i\mathcal{P} = \{A_i x \mid x \in \mathcal{P}\}$, we have

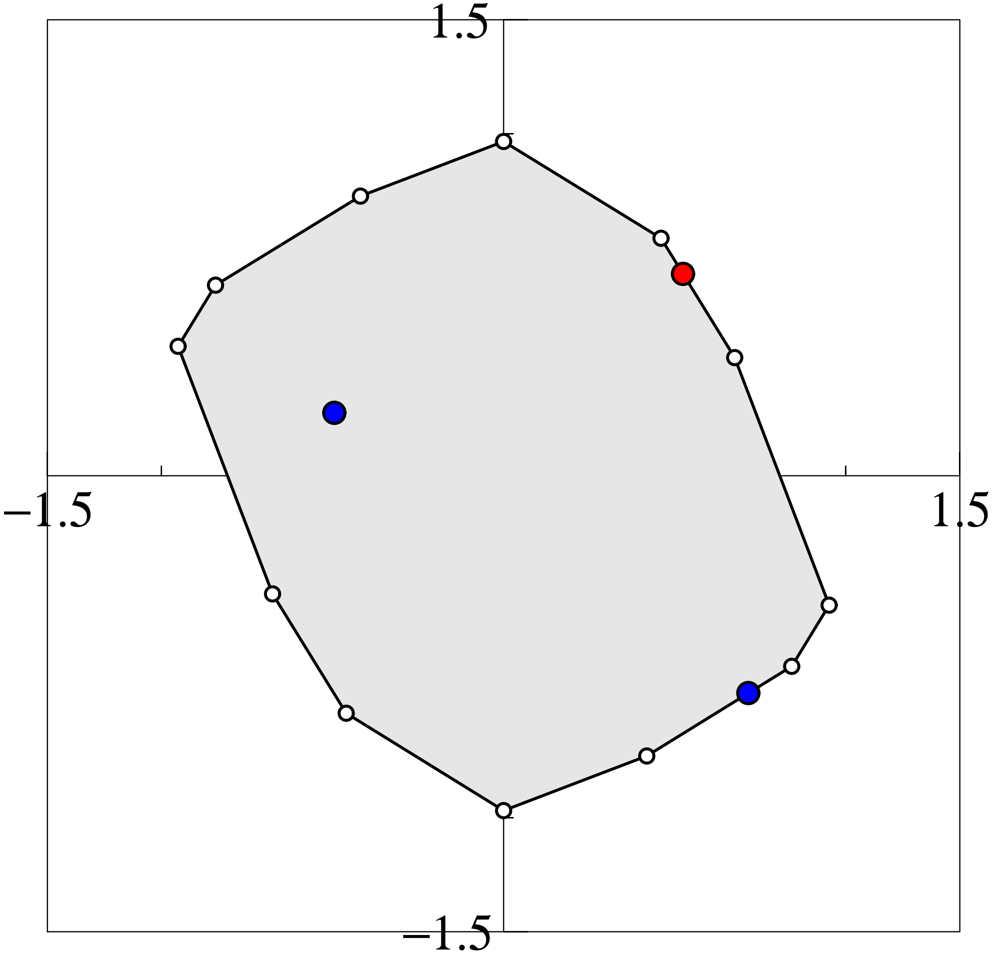
$$\mathcal{P} = \mathrm{convhull}\left(\bigcup_{i=1}^{m} A_i \mathcal{P}\right) \qquad \textbf{(H)}$$

Theorem (**Plinschke & Wirth '08**)
Let $\mathcal{P}$ define an extremal norm $\|\cdot\|_{\mathcal{P}}$ for $\mathcal{F}$ and assume that **(H)** holds. Then $\|\cdot\|_{\mathcal{P}^*}$ is a Barabanov norm for $\mathcal{F}^{\mathrm{T}}$.

Recipe: Given $\mathcal{F}$, apply the polytope algorithm to $\mathcal{F}^{\mathrm{T}}$.

Example 1 (again)

Consider the family $\mathcal{F}^{\mathrm{T}} = \{A_1^{\mathrm{T}}, A_2^{\mathrm{T}}\}$ and the norm $\|\cdot\|_{\mathcal{P}^*}$.
Then we have



For any initial vector $x \in \partial\mathcal{P}^*$ (in red), at least one of the vectors $A_1^{\mathrm{T}}x, A_2^{\mathrm{T}}x \in \partial\mathcal{P}^*$ (in blue).

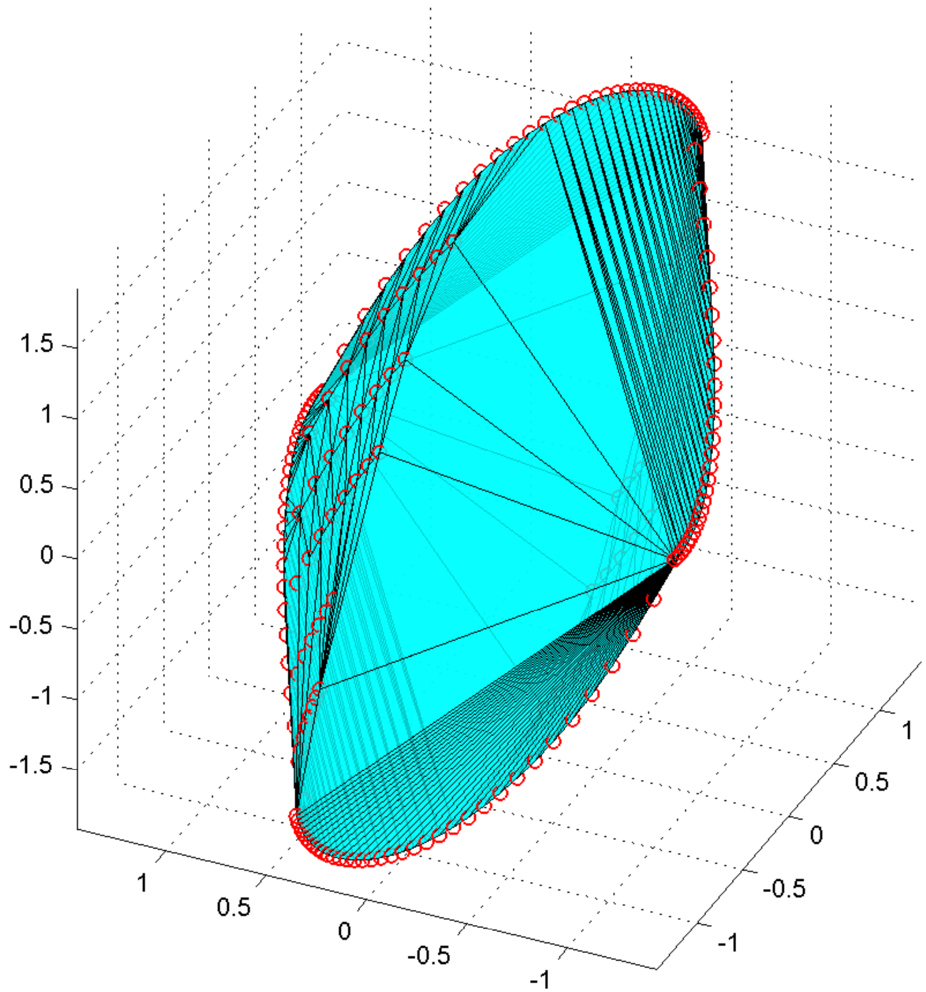
Example in control theory

Consider the 3-dimensional control system

$\dot{x}(t) = B(u(t))\, x(t)$, with

$$B_1 = \begin{pmatrix} -1 & 0 & 0 \\ 10 & -1 & 0 \\ 0 & 0 & -10 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} -10 & 0 & 10 \\ 0 & -10 & 0 \\ 0 & 10 & -1 \end{pmatrix}$$



and discretize it on a grid with $\Delta t = 1/256$.

A MUSL is computed through the determination of a the Barabanov norm whose unit ball $\mathcal{B}$ is shown in the figure.

Software

Matlab routines are made available at

`http://univaq.it/~guglielm/`

THANK YOU