Splitting the differential Riccati equation

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Outline

Splitting methods for evolution equations

The differential Riccati equation

Implementation issues and results

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Abstract evolution equations

We consider

$$\dot{u} = \big(F + G\big)u, \qquad u(0) = u_0$$

on a Hilbert or Banach space H

F typically (nonlinear) diffusion operator

G (nonlinear) perturbation with sufficiently good properties

Splitting methods

Full problem

$$\dot{u} = \big(F + G\big)\,u, \qquad u(0) = u_0$$

difficult and/or expensive

Instead cleverly combine solutions to sub-problems

$$\dot{u} = F\,u \qquad \text{and} \qquad \dot{u} = G\,u$$

Simple and/or cheaper

Splitting methods: examples

$$\mathcal{L}_h = \mathrm{e}^{hF}\mathrm{e}^{hG} \qquad \text{Lie}$$

$$\mathcal{S}_h = \mathrm{e}^{h/2G}\mathrm{e}^{hF}\mathrm{e}^{h/2G} \qquad \text{Strang}$$

$$\mathcal{M}_h = (I - hF)^{-1}\mathrm{e}^{hG} \qquad \text{Mixed Lie}$$

Solution approximations to $u(nh)$ are

$$\mathcal{L}_h^n u_0, \qquad \mathcal{S}_h^n u_0, \qquad \mathcal{M}_h^n u_0$$

Dissipative setting

Dissipative problems (Hilbert space)

$$(Fu - Fv, u - v)_H \leq 0 \qquad u, v \in \mathcal{D}(F)$$

Further

$$\mathcal{D}(F) \subset C, \qquad C \quad \text{closed, convex}$$

with range condition

$$C \subset \mathcal{R}(I - hF), \qquad h \geq 0$$

Consequences

Non-expansive resolvent

$$\|(I - hF)^{-1}u - (I - hF)^{-1}v\|_H \leq \|u - v\|_H$$

$\Rightarrow$ will give stability for the schemes

Semigroup

$$\mathrm{e}^{tF}u_0 = \lim_{n\to\infty}(I - hF)^{-n}u_0, \quad t = nh$$

is mild solution to $\dot{u} = Fu$ if $u_0 \in \overline{\mathcal{D}(F)}$

$\Rightarrow$ existence of solutions, but no further regularity

Splitting analysis

Convergence of various splitting schemes:

Brézis & Pazy 1972, Barbu 1976, etc.

But how fast? Convergence orders

Basic result, Crandall & Liggett 1971:

$$\left\|(I - hF)^{-n} u_0 - u(nh)\right\|_H \leq Ch^p \qquad p \in [1/2, 1]$$

Note: implicit Euler discretization for $\dot{u} = Fu$!

Analysis idea

Stability from dissipativity

Consistency by estimating distance to implicit Euler, $\mathcal{O}(h^q)$

Convergence of order $\mathcal{O}(h^p + h^q)$

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The differential Riccati equation (DRE)

$$\dot{P}(t) = A^* \circ P(t) + P(t) \circ A + Q - P(t) \circ P(t),$$

$$P(0) = P_0$$

Solve for operator-valued $P(t)$ for $t \in [0, T]$

Assumption: Q and P_0 self-adjoint, positive semi-definite

Important in e.g. optimal control - LQR problems

DRE splitting

$$\dot{P}(t) = A^* \circ P(t) + P(t) \circ A + Q - P(t) \circ P(t)$$

$$= \mathcal{F}P + \mathcal{G}P$$

Mixed Lie splitting scheme:

$$\mathcal{M}_h = (I - h\mathcal{F})^{-1} \mathrm{e}^{h\mathcal{G}}$$

More problems, but simpler

Nonlinear subproblem:

$$\mathrm{e}^{h\mathcal{G}} P_0 = (I + hP_0)^{-1} P_0$$

Affine subproblem:

$$U = (I - h\mathcal{F})^{-1} P_0$$

equivalent to

$$(I - 2hA)^* U + U(I - 2hA) = 2hQ + 2P_0$$

(Linear) Lyapunov equation for U

The dissipative setting: Which space?

Given Gelfand triple

$$V \hookrightarrow H \cong H^* \hookrightarrow V^*$$

A, $A^* \in \mathcal{L}(V, V^*)$ given by

$$\langle -Au, v\rangle_{V^* \times V} = a(u, v)$$

$$\langle -A^* u, v\rangle_{V^* \times V} = a(v, u)$$

with bounded, coercive a:

$$|a(u, v)| \leq C_1 \|u\|_V \|v\|_V \quad \text{and} \quad a(u, u) \geq C_2 \|u\|_V^2$$

The dissipative setting: Which space?

Given Gelfand triple

$$V := H_0^1(\Omega) \hookrightarrow L^2(\Omega) \cong L^2(\Omega)^* \hookrightarrow H^{-1}(\Omega) =: V^*$$

A, $A^* \in \mathcal{L}(H_0^1, H^{-1})$ given by

$$\begin{aligned}\langle -Au, v\rangle_{V^*\times V} &= (\sqrt{\alpha}\,\nabla u, \sqrt{\alpha}\,\nabla v)_{L^2} + \lambda(u, v)_{L^2}\\ &= \langle -A^* u, v\rangle_{V^*\times V}\end{aligned}$$

Diffusion operator

$$A = \nabla \cdot \big(\alpha \nabla u\big) - \lambda I$$

Temam setting: Hilbert-Schmidt operators

We look for $P(t)$ in

$$\mathcal{H} = \mathcal{HS}(H, H)$$

Stronger than bounded linear operator $\mathcal{L}(H, H)$

Hilbert space with

$$(f, g)_{\mathcal{H}} = \sum_{k=1}^{\infty} (fe_k, ge_k)_H$$

Why Hilbert-Schmidt?

New Gelfand triple

$$\mathcal{V} \hookrightarrow \mathcal{H} \cong \mathcal{H}^* \hookrightarrow \mathcal{V}^*$$

The operators $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{F} + \mathcal{G}$ with

$$\mathcal{F}P = A^* \circ P + P \circ A + Q \qquad \mathcal{G}P = -P \circ P$$

$$\mathcal{D}(\mathcal{F}) = \{P \in \mathcal{V} \; ; \; \mathcal{F}P \in \mathcal{H}\} \qquad \mathcal{D}(\mathcal{G}) = \mathcal{C}$$

where

$$\mathcal{C} = \{P \in \mathcal{H} \; ; \; P = P^* \; ; \; (Px, x) \geq 0\}$$

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$$\mathcal{C} = \{P \in \mathcal{H}\ ;\ P = P^*\ ;\ (Px, x) \geq 0\}$$

are all dissipative:

$$(\mathcal{F}P_1 - \mathcal{F}P_2, P_1 - P_2)_{\mathcal{H}} \leq 0$$

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where

$$\mathcal{C} = \{P \in \mathcal{H} \; ; \; P = P^* \; ; \; (Px, x) \geq 0\}$$

all satisfy the range condition:

$$(I - h\mathcal{F})^{-1}\mathcal{C} \subset \mathcal{C}$$

Convergence order

Mixed Lie

$$\mathcal{M}_h = (I - h\mathcal{F})^{-1} \mathrm{e}^{h\mathcal{G}}$$

Stability by dissipativity of $\mathcal{F}$, $\mathcal{G}$

Consistency for $P_0 \in \mathcal{D}(\mathcal{F}) \cap C$

$$\left\| \mathcal{M}_h^n P_0 - (I - h(\mathcal{F} + \mathcal{G}))^{-n} P_0 \right\|_{\mathcal{H}} \leq Ch$$

Convergence of order $\mathcal{O}(h^p + h)$ $\quad$ ($p =$ IE order)

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Large-scale setting

Solving for large (dense) matrices:

$$P(t) \in \mathbb{R}^{N \times N} \quad \text{with} \quad N \in [10^3, 10^7]$$

Storage?

Essential to use structural properties!

Low-rank: $\quad P \approx zz^\mathsf{T} \quad$ with $z \in \mathbb{R}^{N \times m}$

Our splitting methods preserve low rank: $\mathcal{M}_h zz^\mathsf{T} = ww^\mathsf{T}$

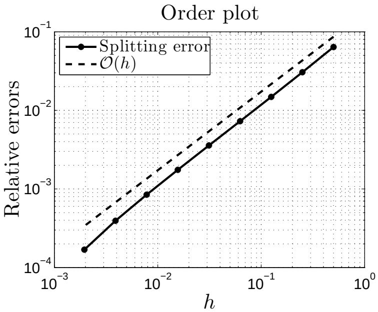
Results: Mixed Lie

$$\dot{P}(t) = A^{\mathsf{T}} P(t) + P(t)A + C^T C - P(t)P(t)$$

$A = \nabla \cdot \big(\alpha \nabla u\big) - I,$ per. BC

$\alpha(x) = 2 + 2\cos 2\pi x$

$$C \sum_{k=0}^{\infty} a_k \mathrm{e}^{2\pi i k x} = \sum_{k=0}^{4} a_k \mathrm{e}^{2\pi i k x}$$



Discretization: 2001 points in space

Relative errors, $\|\mathcal{M}_h^n P_0 - P(nh)\|_{\text{Fro}} \,/\, \|P(nh)\|_{\text{Fro}}$

Thank you

References on my webpage:
http://www.maths.lu.se/staff/tony-stillfjord/research/