Analysis of underdamped linear systems driven by Brownian motion with the help of ARMA processes

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1. Theoretical basics
2. Application in structural health monitoring
3. A case study

1. Theoretical basics

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A linear SDE

- We consider the SDE:

$$\dot{Y}(t) = AY(t) + B\dot{W}_t$$

- Underdamped: all eigenvalues of $A \in \mathbb{R}^{n\times n}$ have negative real parts

$$\rightarrow Y(t) = \int_{-\infty}^{t} \mathrm{e}^{A(t-s)} B \,\mathrm{d}W_s$$

  is stationary.
- The components of the uniformly sampled solution $X_k := Y_i(\tau k)$ are stationary time series.

ARMA processes

Definition

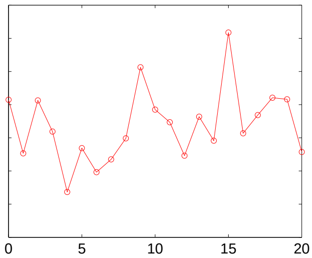
A stationary process $\{X_t, t \in \mathbb{Z}\}$ with mean zero is called an ARMA(p, q) process if for every $t \in \mathbb{Z}$ it holds that

$$X_t - \varphi_1 X_{t-1} - \ldots - \varphi_p X_{t-p} = a_t + \vartheta_1 a_{t-1} + \ldots + \vartheta_q a_{t-q},$$

where $\{a_t\} \sim \mathrm{WN}(0, \sigma^2)$.

Notation:

- $\varphi(z) := 1 - \varphi_1 z - \ldots - \varphi_p z^p$
- $\vartheta(z) := 1 + \vartheta_1 z + \ldots + \vartheta_q z^q$
- $SX_t = X_{t-1}$
- $\varphi(S)X_t = \vartheta(S)a_t$



Coherence

- Let $X_k := Y_i(\tau k)$ be a component of the uniformly sampled solution of $Y(t) = \int_{-\infty}^{t} \mathrm{e}^{A(t-s)} B \,\mathrm{d} W_s$.
- Then X_k is covariance equivalent to an ARMA(p, q) process with $n = p > q$.
- There is a relation of the eigenvalues of A and the autoregressive coefficients. This relation can be used to determine eigenvalues of underdamped systems.

Model for engineering structures

A model for viscously damped engineering structures:

$$M\ddot{Z}(t) + R\dot{Z}(t) + KZ(t) = u\dot{W}(t), \qquad (1)$$

$Z(t)$ ... displacement
M, R, K ... mass, damping and stiffness properties
$u\dot{W}(t)$ ... random excitation

The state space system

$$\dot{Y}(t) = AY(t) + B\dot{W}_t$$

with

$$Y = \begin{pmatrix} Z \\ \dot{Z} \end{pmatrix}, \; A = \begin{pmatrix} 0 & I \\ M^{-1}R & M^{-1}K \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 \\ u \end{pmatrix}$$

usually forms an underdamped system and thus an ARMA process when sampled.

Example

The uniformly sampled (with samplingrate τ) solution of the second-order system

$$m\ddot{Z}(t) + r\dot{Z}(t) + kZ(t) = \dot{W}_t$$

is covariance equivalent to an ARMA$(2,1)$ process with

$$\varphi_1 = 2\mathrm{e}^{-r\tau/2m} \cos\left(\tau\sqrt{\frac{k}{m} - \frac{r^2}{4m^2}}\right),$$

$$\varphi_2 = -\mathrm{e}^{r\tau/m}.$$

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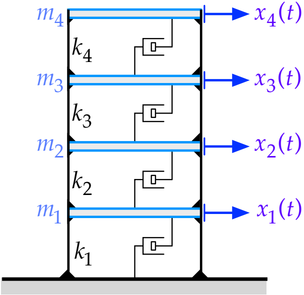
Aluminium shear frame

- small-scale four storey shear frame
- two horizontal columns of a 15 $\times$ 2 mm cross section
- four horizontal beams of 19 $\times$ 19 mm cross section
- affixed on a 250kg concrete block

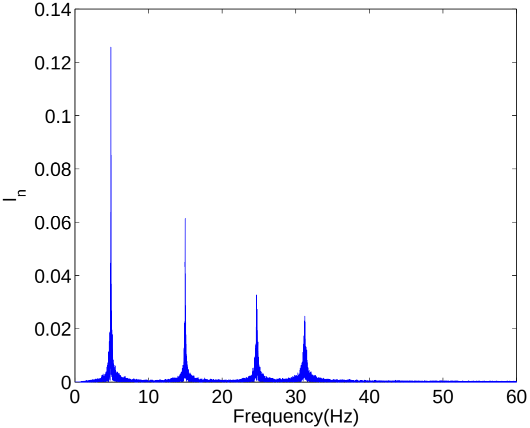
Aluminium shear frame cont.

- vibrations out of plane neglected
- beams are considered to be rigid
- → the displacement of each storey expressed by 1 DoF
- → 4 DoF model for the whole structure



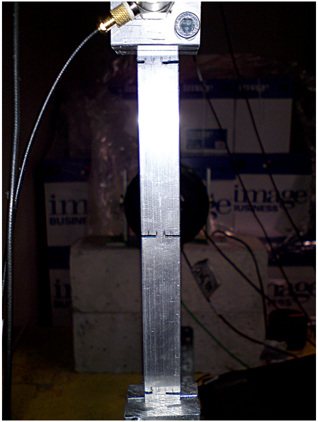
Aluminium shear frame cont.

⇒ 4 dominant modes of a measurement stream



Damage states

- damage conducted by cut into vertical columns
- 13 different damage states
- incision of 3mm at a single location between two consecutive damage states



A distance measure

Consider the mean squared error as sum of one-step prediction errors:

$$\mathrm{MSE}(\varphi, \vartheta, x) := \frac{1}{n-1} \sum_{t=1}^{n} \left(\frac{\vartheta(S)}{\varphi(S)} x_t \right)^2$$

(remember $\varphi(z) := 1 - \varphi_1 z - \ldots - \varphi_p z^p$, $\vartheta(z) := 1 + \vartheta_1 z + \ldots + \vartheta_q z^q$) and $\{a_t\} \sim \mathrm{WN}(0, \sigma^2)$)

Define the distance between two measurements x and $\widetilde{x}$:

1. Estimate the ARMA coefficients $(\varphi, \vartheta, \sigma)$ of x and $\left(\widetilde{\varphi}, \widetilde{\vartheta}, \widetilde{\sigma}\right)$ of $\widetilde{x}$
2. Normalize the streams by $x := x/\sigma$ and $\widetilde{x} := \widetilde{x}/\widetilde{\sigma}$
3. Set the distance as $d\left(x, \widetilde{x}\right) := \min\left(\mathrm{MSE}(\varphi, \vartheta, \widetilde{x}), \mathrm{MSE}(\widetilde{\varphi}, \widetilde{\vartheta}, x)\right)$

Empiric threshold

Compare the distance of six measurements of the reference state using an ARMA$(8,7)$ process:

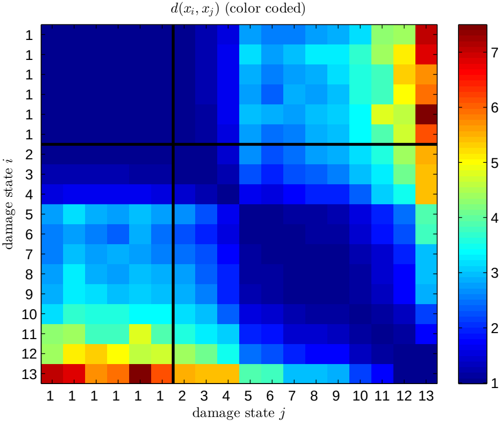
No.	**1**	**2**	**3**	**4**	**5**	**6**
1	1.00	1.02	1.05	1.04	1.01	1.03
2	1.02	1.00	1.01	1.02	1.02	0.99
3	1.05	1.01	1.00	1.01	1.01	1.02
4	1.04	1.02	1.01	1.00	1.00	1.02
5	1.01	1.02	1.01	1.00	1.00	0.99
6	1.03	0.99	1.02	1.02	0.99	1.00

$\rightsquigarrow$ streams with a distance greater than 1.05 are considered as different else considered as equal

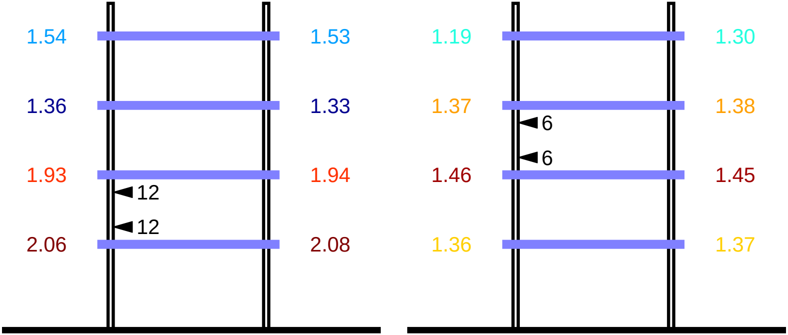
Performance

Comparing 6 measurements of the reference state with 13 different damage state measurements:

- 133 compared pairs of different states are identified as different
- 5 compared pairs of different states are identified as equal



Damage location



Conclusion and Outlook

- ☑ relation between ARMA coefficients and system modes
- ☑ distance measure to detect structural changes
- ☑ damage location worked in experiments
- ☐ theoretical analysis of damage location methods
- ☐ automatical choice of process order and samplingrate

Thank you for your attention!