Modification of dimension splitting methods for two dimensional parabolic problems and its limitations in higher dimensions

Tobias Hell

joint work with Alexander Ostermann & Bertram Tschiderer

Numerical Analysis of Evolution Equations
Innsbruck, 15th of October, 2014

Setting

Evolution equation. $u'(t) = \mathcal{L}u(t) + g(t), \quad u(0) = u_0$

Strictly elliptic operator. $\mathcal{L} = \partial_x(a\partial_x) + \partial_y(b\partial_y)$

Domain. $\Omega = (0,1)^2$, $\quad D(\mathcal{L}) = H^2(\Omega) \cap H_0^1(\Omega)$

Coefficients. $a, b \in \mathcal{C}^2(\overline{\Omega})$ positive and bounded away from 0

Resolvent splitting

$$u'(t) = \mathcal{L}u(t) + g(t), \quad u(0) = u_0 \qquad \mathcal{L} = \partial_x(a\partial_x) + \partial_y(b\partial_y)$$

Time stepping. $\tau > 0$, $t_n = n\tau$ for $n \in \mathbb{N}_0$

Implicit Euler. $u(t_{n+1}) \approx u_{n+1} = (I - \tau\mathcal{L})^{-1}(u_n + \tau g(t_n))$

Split operators. $\mathcal{A} = \partial_x(a\partial_x)$, $\mathcal{B} = \partial_y(b\partial_y) \Rightarrow \mathcal{L} = \mathcal{A} + \mathcal{B}$

Lie resolvent splitting.

$$u_{n+1} = \underbrace{(I - \tau\mathcal{B})^{-1}(I - \tau\mathcal{A})^{-1}}_{\approx (I-\tau\mathcal{L})^{-1}}(u_n + \tau g(t_n))$$

Order of the Lie resolvent splitting

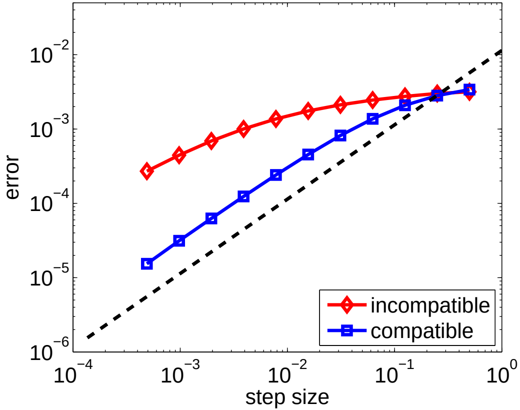

Figure: Discrete L^2 error of the Lie resolvent splitting applied to two parabolic problems on $(0,1)^2$. The dashed line has slope 1.

Pointwise error of the Lie resolvent splitting

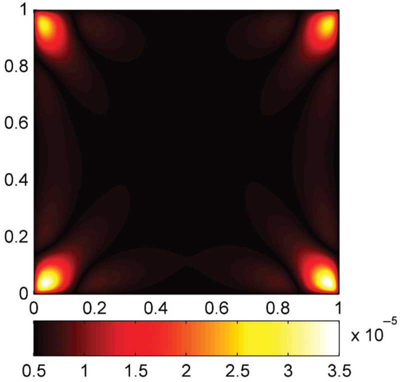

Figure: Pointwise absolute error of the Lie resolvent splitting applied to a parabolic problem on $(0,1)^2$.

Source of the order reduction

Full convergence order. $\forall\, t \geq 0\colon \mathcal{L}^{-1}g(t) \in H^4(\Omega)$

Stationary problem. $f := g(t)$ for fixed $t \geq 0$

$$\begin{cases} \mathcal{L}w = f & \text{on } \Omega, \\ w|_{\partial\Omega} = 0 \end{cases}$$

A. Ostermann, K. Schratz: *Error analysis of splitting methods for inhomogeneous evolution equations.* Appl. Numer. Math. (2012)

Source of the order reduction

Full convergence order. $\forall\, t \geq 0\colon \mathcal{L}^{-1} g(t) \in H^4(\Omega)$

Stationary problem. $f := g(t)$ for fixed $t \geq 0$

$$\begin{cases} \Delta w = f & \text{on } \Omega, \\ w|_{\partial\Omega} = 0 \end{cases}$$

Question. For given $f \in H^{2k}(\Omega)$, does $w \in H^{2k+2}(\Omega)$ hold?

Answer. No – not in general.

Compatibility conditions. $V \ldots$ set of vertices of $(0,1)^2$

$$w \in H^{2k+2}(\Omega) \iff \sum_{i=1}^{j} (-1)^{i+1} \partial_x^{2j-2i} \partial_y^{2i-2} f\big|_V = 0, \; j = 1, \ldots, k$$

T. H., A. Ostermann: *Compatibility conditions for Dirichlet and Neumann problems of Poisson's equation on a rectangle*. J. Math. Anal. Appl. (2014)

Modifcation of the Lie resolvent splitting

Order reduction. $w \in H^4(\Omega) \iff f|_V = 0$
$g(t) \neq 0$ at corners $\Rightarrow$ order $1/4 - \varepsilon$ (*incompatible*)

Modification. Assume $g(0)(0,0) \neq 0$.

1. Solve the stationary problem
$$\mathcal{L}w = \tfrac{g(0)}{g(0)(0,0)} \text{ in } \Omega, \quad w|_{\partial\Omega} = 0.$$
2. Full convergence order of 1. Lie resolvent splitting applied to
$$u_c'(t) = \mathcal{L}u_c(t) + g_c(t), \quad u_c(0) = u_0 + g(0)(0,0)w$$
with $g_c(t) := g(t) - g(t)(0,0)\tfrac{g(0)}{g(0)(0,0)} + \partial_t g(t)(0,0)w$.
3. Superposition. $u(t) = u_c(t) - g(t)(0,0)w$

T. H., A. Ostermann, M. Sandbichler: *Modification of dimension splitting methods – overcoming the order reduction due to corner singularities*. To appear in: IMA J. Numer. Anal.

Pointwise error of the Lie resolvent splitting

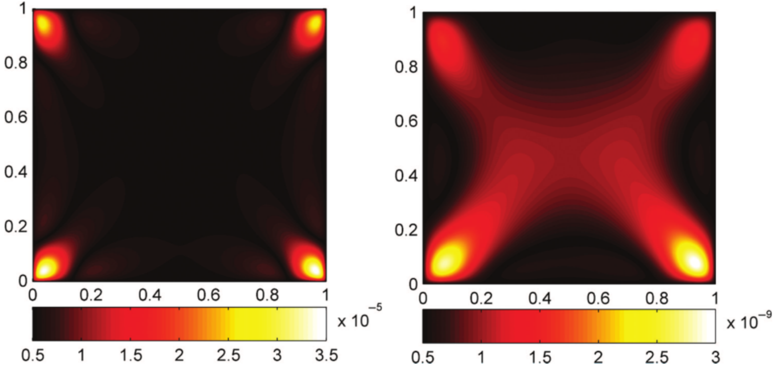

Figure: Pointwise error of the standard (left) and the modified (right) Lie resolvent splitting applied to a parabolic problem on $(0,1)^2$.

Order of the Lie resolvent splitting

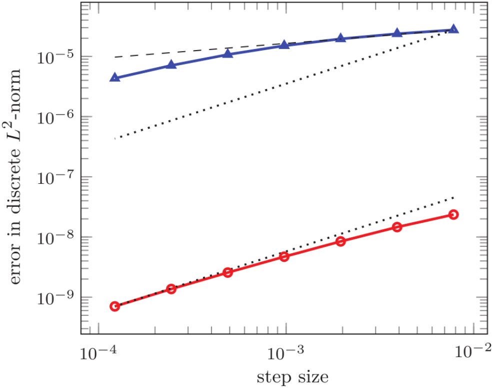

Figure: Discrete L^2 error of the standard (blue triangles) and the modified Lie resolvent splitting (red circles). The dashed and dotted lines have slope 1/4 and 1, respectively.

Pointwise error of the exponential Strang splitting

Exponential Strang splitting.

$\mathrm{e}^{\tau\mathcal{L}} \approx$ compositions of $\mathrm{e}^{\tau\mathcal{A}}$ and $\mathrm{e}^{\tau\mathcal{B}}$

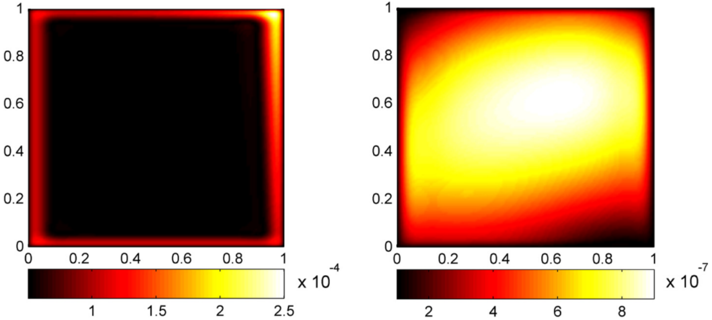

Figure: Pointwise absolute error of the standard (left) and the modified (right) exponential Strang splitting.

E. Faou, A. Ostermann, K. Schratz: *Analysis of exponential splitting methods for inhomogeneous parabolic equations*. To appear in: IMA J. Numer. Anal.

Order of the exponential Strang splitting

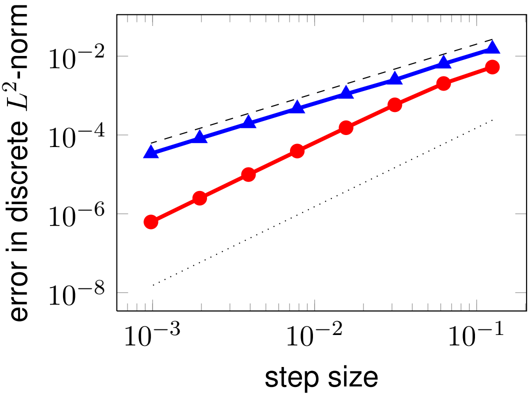

Figure: The discrete L^2-error of the modified (red circles) and the classical (blue triangles) exponential Strang splitting. The dashed and dotted lines have slope 1.25 and 2, respectively.

E. FAOU, A. OSTERMANN, K. SCHRATZ: *Analysis of exponential splitting methods for inhomogeneous parabolic equations*. To appear in: IMA J. Numer. Anal.

Order of the Lie resolvent splitting in 3 dimensions

Order reduction of the Lie resolvent splitting in 3 dimensions

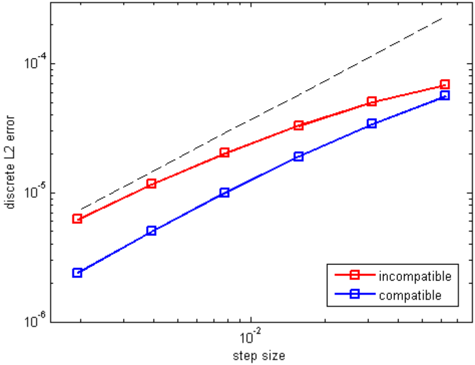

Figure: Discrete L^2 error of the Lie resolvent splitting applied to two parabolic problems on $(0,1)^3$. The dashed line has slope 1.

Source of the order reduction in higher dimensions

Stationary problem. $\Omega = (0,1)^d$ for $d \in \mathbb{N}$ with $d > 2$

$$\begin{cases} \Delta w = f \quad \text{on } \Omega, \\ w|_{\partial\Omega} = 0 \end{cases}$$

Question. For given $f \in H^2(\Omega)$, does $w \in H^4(\Omega)$ hold?

Answer. No – not in general.

Compatibility conditions. $E \ldots$ set of edges of $(0,1)^d$

$$w \in H^4(\Omega) \iff f|_E = 0 = \partial_j f|_E,\ j = 1, \ldots, d$$

Question. Similar modification as in two dimensions ”possible”?