On a Spatial Epidemic Propagation Model

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Outline

Epidemic models

Mathematical models of diseases

Basic assumptions.

- the population is presumed to be constant in size;
- they are divided into three classes:
  1. I(t): infected individuals who can pass on the disease to others;
  2. S(t): susceptibles who have yet to contract the disease and become infectious,
  3. R(t): members who have been infected but cannot transmit the disease for some reason, e.g., they have been isolated from the rest of the population.

Mathematical models of diseases(cont'd)

Kermack and McKendrick, 1927

$$S' = -aSI,$$
$$I' = aSI - bI,$$
$$R' = bI,$$

$I = I(t)$: number of infective,
$S = S(t)$: number of susceptible and
$R = R(t)$: number of recovered (removed) members.

$a > 0$: contact rate; $b > 0$: recovery coefficient

This model has been improved several times taking into the account also *births, deaths, latent periods, reinfections, incubations* etc.

Inclusion of spatial dependence

How spatial dependence can be introduced into the model?

- Consider subpopulations that are connected into a network.
- Allow the motion of the individuals in the population.
- We will consider another approach:
  – the speed of the motion of the individuals can be neglected (compared to the speed of the disease)
  – a member of the population can infect only members in its well defined spatial neighbourhood.

Inclusion of spatial dependence

Mathematical model with spatial dependence

$$S'_t(x,t) = -\left(\int_{N(x)} W(|x'-x|) I(x',t)\,\mathrm{d}x'\right) S(x,t),$$

$$I'_t(x,t) = \left(\int_{N(x)} W(|x'-x|) I(x',t)\,\mathrm{d}x'\right) S(x,t) - bI(x,t),$$

$$R'_t(x,t) = bI(x,t),$$

$S = S(x,t)$, $I = I(x,t)$ and $R = R(x,t)$ are now spatial dependent densities.

The nonnegative weighting function W depends only on the distance of the points x' and x, and $N(x)$ is a prescribed neighbourhood of the point x.

Inclusion of spatial dependence

Simplifications [Jones, Sleeman, 2011]:

- We consider 1D problems.
- $N(x) = [x - \delta, x + \delta]$.
- I is approximated with its second order spatial Taylor series.

Simplified mathematical model with spatial dependence

$$\begin{aligned} S'_t &= -S\left(\theta I + \phi I''_{xx}\right), \\ I'_t &= S\left(\theta I + \phi I''_{xx}\right) - bI, \\ R'_t &= bI, \end{aligned}$$

where

$$\theta = \int_{-\delta}^{\delta} W(|u|)\,\mathrm{d}u, \quad \phi = \frac{1}{2}\int_{-\delta}^{\delta} u^2 W(|u|)\,\mathrm{d}u$$

are positive constants that can be computed from the model.

Properties of the mathematical model

Some qualitative properties of the model

Simplified mathematical model with spatial dependence

$$S'_t = -S\left(\theta I + \phi I''_{xx}\right),$$
$$I'_t = S\left(\theta I + \phi I''_{xx}\right) - bI,$$
$$R'_t = bI.$$

Our requirements are:

P1 Additivity property
$S + I + R$ is constant at a fixed spatial position.

P2 Monotonicity property
– S monotone decreases in time
– R monotone increases in time

P3 Nonnegativity property
$S > 0$, $I \geq 0$, $R \geq 0$ at $t = 0$
$\Downarrow$
$S, I, R \geq 0.$

Some qualitative properties of the model

Lemma. If the condition

$$\theta I + \phi I''_{xx} \geq 0 \tag{1}$$

is satisfied then properties [P2] and [P3] are true. Property [P1] is true without any restrictions.

Remark. The condition (1) is also necessary.

Remark. The above condition is hardly checkable: it depends on the values of the solution I in the whole solution domain.

Travelling wave solutions

In order to model epidemic waves we are looking for travelling wave solutions. Let us set

$$S(x,t) = \tilde{S}(x-ct), \quad I(x,t) = \tilde{I}(x-ct), \quad R(x,t) = \tilde{R}(x-ct),$$

where c is the constant wave speed, $\tilde{I}$ and $\tilde{S}$ have the properties

$$\lim_{\xi\to\pm\infty} \tilde{I}(\xi) = 0, \quad \lim_{\xi\to\pm\infty} \tilde{I}'(\xi) = 0, \quad \lim_{\xi\to\infty} \tilde{S}(\xi) = \tilde{S}^{\infty} > 0.$$

Travelling wave solutions

After some manipulations we get:

Form of the system after inserting the wave form solutions

$$\tilde{S}' = \frac{c}{\phi}\log(\tilde{S}/\tilde{S}^{\infty}) - \frac{\theta c}{b\phi}(\tilde{I} + \tilde{S} - \tilde{S}^{\infty}),$$

$$\tilde{I}' = \frac{b}{c}\tilde{I} - \frac{c}{\phi}\log(\tilde{S}/\tilde{S}^{\infty}) + \frac{\theta c}{b\phi}(\tilde{I} + \tilde{S} - \tilde{S}^{\infty}),$$

$$\tilde{R}' = -\frac{b}{c}\tilde{I}.$$

Travelling wave solutions

Let us introduce the notations

$$\tilde{S}^{-\infty} = \lim_{\xi \to -\infty} \tilde{S}(\xi), \ \tilde{S}^{\infty} = \lim_{\xi \to \infty} \tilde{S}(\xi).$$

Lemma. The necessary condition of the travelling wave solution is

$$\tilde{S}^{\infty} > b/\theta.$$

Moreover

$$\tilde{S}^{-\infty} < b/\theta,$$

that is the epidemic wave does not leave enough susceptible members back to be able to sustain a new wave. If the necessary condition is satisfied then

$$c \geq 2\sqrt{\tilde{S}^{\infty}\phi(\tilde{S}^{\infty}\theta - b)}$$

is a lower bound for the wave speed.

Numerical solution and its properties

Finite difference scheme

We define a uniform spatial grid $\omega_h = \{x_k \in [0, L] \,|\, x_k = kh,\ k = 0, \ldots, N,\ h = L/N\}$ and a positive time step $\tau > 0$.

We apply the notations $s_k^n \approx S(kh, n\tau)$, etc.

We define the difference scheme as follows.

$$\frac{s_k^{n+1} - s_k^n}{\tau} = -s_k^n \left(\theta i_k^n + \phi \frac{i_{k-1}^n - 2i_k^n + i_{k+1}^n}{h^2} \right),$$

$$\frac{i_k^{n+1} - i_k^n}{\tau} = s_k^n \left(\theta i_k^n + \phi \frac{i_{k-1}^n - 2i_k^n + i_{k+1}^n}{h^2} \right) - b i_k^n,$$

$$\frac{r_k^{n+1} - r_k^n}{\tau} = b i_k^n,$$

$k = 0, \ldots, N$, $s_{-1}^n = s_1^n$, $s_{N+1}^n = s_{N-1}^n$, etc.

Properties of the numerical solution

Lemma. The finite difference scheme satisfies the discrete version of [P1], and if the relation

$$0 \leq \theta i_k^n + \phi \frac{i_{k-1}^n - 2i_k^n + i_{k+1}^n}{h^2}$$

is true for all possible indices k and n, and the time step satisfies the condition

$$\tau \leq \min \left\{ \frac{1}{M(\theta + 4\phi/h^2)}, 1/b \right\},$$

where $M = \max_{x\in[0,L]}\{S(x,0) + I(x,0) + R(x,0)\}$, then the discrete versions of the properties [P2] and [P3] are also satisfied. Under the above conditions the scheme is stable in maximum norm.

Properties of the numerical solution (cont'd)

The condition

$$0 \le \theta i_k^n + \phi \frac{i_{k-1}^n - 2i_k^n + i_{k+1}^n}{h^2}$$

is an a'posteriori condition. How to guaranty it a'priori?

We introduce the notation:

$$j_k^n := \theta i_k^n + \phi \frac{i_{k-1}^n - 2i_k^n + i_{k+1}^n}{h^2}.$$

Suppose that there are positive numbers P, Q, p, $q > 0$ such that

$$Q|u - \delta|^q \le W(u) \le P|u - \delta|^p, \text{ if } u \in [0, \delta]. \qquad (2)$$

Properties of the numerical solution (cont'd)

Lemma. Assume that we have

$$\tau \leq \min\left\{\frac{1}{M(\theta + 4\phi/h^2)}, 1/b\right\},$$

or

$$h \geq \sqrt{2P/Q}\,\frac{\delta^{p/2-q/2+1}\sqrt{q+1}}{\sqrt{(p+1)(p+2)(p+3)}}$$

and the time-step satisfies

$$\tau \leq \min\left\{\frac{1}{M(\theta + 4\phi/h^2)}, 1/\left(b + \frac{2\phi M}{h^2}\right)\right\}.$$

Then the nonnegativity of i_k^n, s_k^n, r_k^n and j_k^n imply the nonnegativity of the next approximations $i_k^{n+1}, s_k^{n+1}, r_k^{n+1}$ and j_k^{n+1}.

Properties of the numerical solution (cont'd)

Lemma. Assume that we have

$$h \geq \sqrt{2P/Q}\,\frac{\delta^{p/2-q/2+1}\sqrt{q+1}}{\sqrt{(p+1)(p+2)(p+3)}}$$

and

$$\tau \leq \min\left\{\frac{1}{M(\theta+4\phi/h^2)}, 1/b\right\},$$

or the time-step satisfies

$$\tau \leq \min\left\{\frac{1}{M(\theta+4\phi/h^2)}, 1/\left(b+\frac{2\phi M}{h^2}\right)\right\}.$$

Then the nonnegativity of i_k^n, s_k^n, r_k^n and j_k^n imply the nonnegativity of the next approximations $i_k^{n+1}, s_k^{n+1}, r_k^{n+1}$ and j_k^{n+1}.

Properties of the numerical solution (cont'd)

Hence we have:

Theorem. Under the conditions of the previous lemma, the nonnegativity of the initial values i_k^0, s_k^0, r_k^0 and j_k^0 imply the discrete versions of the qualitative properties [P2] and [P3], and the scheme is stable in the maximum norm, too.

Remark. For the asymptotic case ($h \to 0$) the condition has the form:

$$\frac{\tau}{h^2} \leq \frac{1}{4\phi M}.$$

Numerical tests

Parameter setting to the numerical tests

Numerical simulation

We solve the problem on the interval [0,10] with standard finite difference schemes.

In the epidemic case the parameters are as follows:

- Model parameters: $b = 0.5$, $\phi =$2.86e-005, $\theta = 0.7$.
- Mesh parameters: $\Delta x = 1/199$, $\Delta t = 1$ $(\Delta t_{\max} = 1.34)$
- Condition of the wave propagation: $b/\theta > 0.71$

Summary

- We have formulated the basic qualitative properties of the continuous and discrete epidemic models.
- We gave necessary and sufficient condition for the continuous model.
- We gave a'priori checkable sufficient condition for the discrete model.
- We analyzed the travelling wave solutions in the continuous models and in the numerical examples.

Future work

- More complex discrete models (not only diffusion).
- $1D \Rightarrow 2D, 3D$.
- Other qualitative properties?
- Other semidiscretization?

Thank you for your attention