Numerical stability for nonlinear evolution equations

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joint work with
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Outline

Original problem

X, Y normed spaces

$F : D \subset X \to Y$

$$F(u) = 0$$

Suppose $\exists\,!\,\bar{u} \in D$ solution

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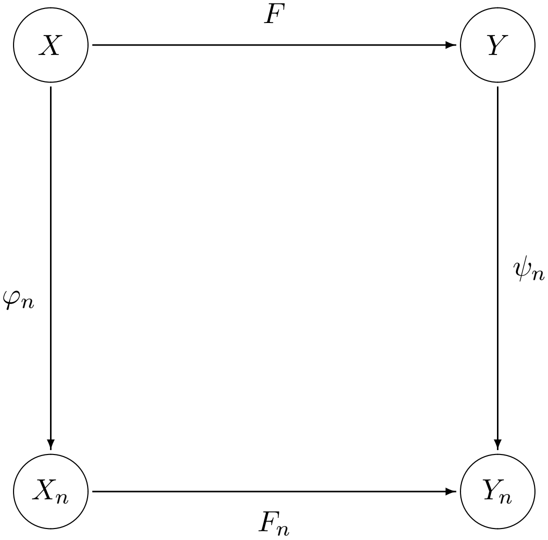
Discrete problem

X_n, Y_n normed spaces

$F_n: D_n \subset X_n \to Y_n$

$$F_n(u_n) = 0, \quad n \in \mathbb{I} = \mathbb{N}^{d+1}$$

Suppose $\exists\,!\,\bar{u}_n \in D_n$ solution


F
X
Y
φ_n
ψ_n
X_n
Y_n
F_n

Definition

The numerical method is called

- *convergent if*

$$\lim_{n\to\infty} \|\varphi_n(\bar{u}) - \bar{u}_n\|_{X_n} = 0$$

- *consistent on* $v \in D$ *if* $\varphi_n(v) \in D_n$ *for some* $n \in \mathbb{I}$ *and*

$$\lim_{n\to\infty} \|F_n(\varphi_n(v)) - \psi_n(F(v))\|_{Y_n} = 0$$

- *N-stable if there exists* $S > 0$ *such that*

$$\|v_n - z_n\|_{X_n} \le S\,\|F_n(v_n) - F_n(z_n)\|_{Y_n}$$

holds for all $v_n, z_n \in D_n$

Proposition

Suppose: F_n *consistent* on $\bar{u}$, *N-stable* , and $\psi_n(0) \xrightarrow{n\to\infty} 0$

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Proof

$$\|\varphi_n(\bar{u}) - \bar{u}_n\|_{X_n} \overset{\text{N-stability}}{\leq} S\|F_n(\varphi_n(\bar{u})) - F_n(\bar{u}_n)\|_{Y_n}$$

$$\leq S \underbrace{\|F_n(\varphi_n(\bar{u})) - \psi_n(F(\bar{u}))\|_{Y_n}}_{\text{consistency}} + S \underbrace{\|\psi_n(F(\bar{u})) - F_n(\bar{u}_n)\|_{Y_n}}_{\text{assumption on } \psi_n(0)} \to 0$$

From now on we consider

$$F(u) := \dot{u} - A(u)$$

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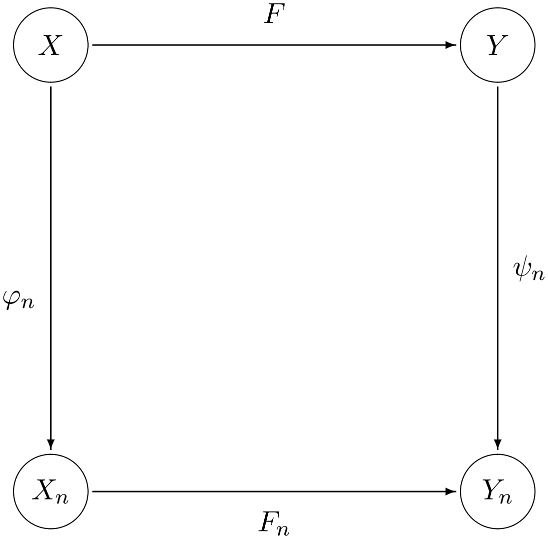
more precisely

$$\big(F(u)\big)(t,x) = \big(\tfrac{\mathrm{d}}{\mathrm{d}t}u(t,\cdot) - A\big(u(t,\cdot)\big)\big)(x)$$

for all $t \geq 0$, $x \in \Omega \subset \mathbb{R}^d$

With what kind of A ?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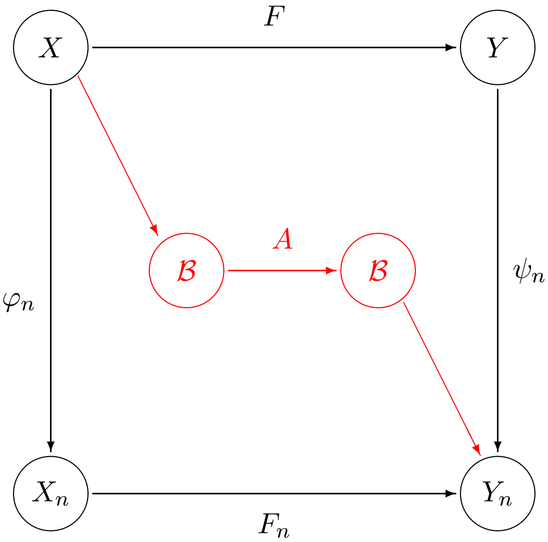
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Assumption on A

Let $\mathcal{B}$ Banach, $A \subset \mathcal{B} \times \mathcal{B}$, $\omega \in \mathbb{R}$

Let A be m-dissipative of type ω, that is:

- $\|(I - \tau A)(v) - (I - \tau A)(z)\|_{\mathcal{B}} \geq (1 - \tau\omega)\|v - z\|_{\mathcal{B}}$

  for all $v, z \in \mathcal{B}$ and $\tau \in \left(0, \frac{1}{|\omega|}\right)$
- $(I - \tau A)$ surjective for all $\tau \in \left(0, \frac{1}{|\omega|}\right)$

Theorem (Crandall–Liggett, 1971)

Let A *be* m*-dissipative of type* $\omega \in \mathbb{R}$ *. Then for all* $v \in \overline{D(A)}$ *:*

$$\exists \lim_{k\to\infty} \left(\left(I - \tfrac{t}{k}A\right)^{-1}\right)^{k}(v) =: S(t)(v)$$

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Properties of S (nonlinear semigroup)

- $S(0) = I$, identity on $\mathcal{B}$
- $S(t+s) = S(t)S(s)$ for all $t, s \geq 0$
- for all $v, z \in \overline{D(A)}$ and $t \geq 0$:

$$\|S(t)(v) - S(t)(z)\|_{\mathcal{B}} \leq \mathrm{e}^{\omega t}\|v - z\|_{\mathcal{B}}$$

- $\lim\limits_{t\searrow 0} S(t)(v) = v$ for all $v \in \overline{D(A)}$

Theorem (1st Brezis–Pazy, 1972)

Let A *be* m*-dissipative of type* ω *and*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} u(t,\cdot) = A\big(u(t,\cdot)\big), & t \geq 0 \\ u(0,\cdot) = u_0(\cdot) \end{cases}$$

Then: $u(t,\cdot) = \big(S(t)(u_0)\big)(\cdot)$ *unique integral solution*

Let $u(0,\cdot) = u_0(\cdot)$ be given. How to approximate $u(t,\cdot) \sim u(t)$?

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SPACE $$\approx S_m\left(\tfrac{t}{k}\right)\left(u\left((k-1)\tfrac{t}{k}\right)\right) \qquad A_m \rightsquigarrow S_m(\cdot)$$

$$\wr\wr$$

TIME $$\approx r\left(\tfrac{t}{k}A_m\right)\left(u\left((k-1)\tfrac{t}{k}\right)\right) \qquad r(\cdot A_m)$$

Space discretisations

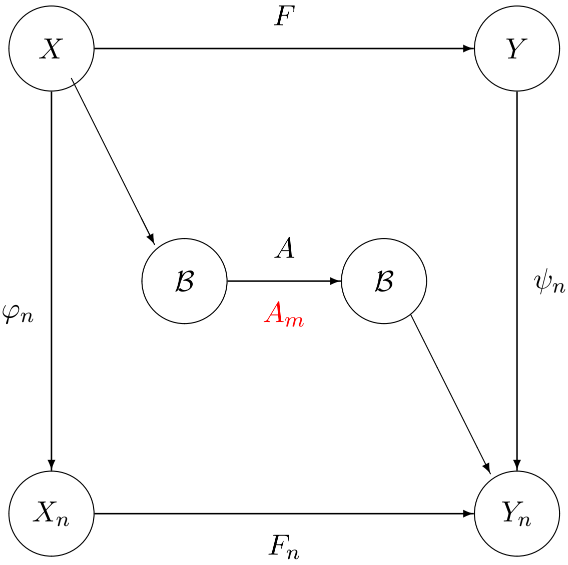
Theorem (2nd Brezis–Pazy, 1972)

Suppose there exist A_m , $m \in \mathbb{N}$ *on* $\mathcal{B}$ *such that*

- A_m *are* m*-dissipative with* ω_m
- $\exists\, \alpha \in [0, \infty)$*:* $0 \le \omega, \omega_m \le \alpha$
- $D(A_m) \supset D(A)$
- $\lim_{m\to\infty} A_m(v) = A(v)$ *for all* $v \in \overline{D(A)}$

Then: $\lim_{m\to\infty} S_m(t)(v) = S(t)(v)$

for all $v \in \overline{D(A)}$ *uniformly for* t *in compact intervals*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Convergence analysis

Consider

$$r\left(\tfrac{t}{k}A_m\right) := \left(I - \tfrac{t}{k}A_m\right)^{-1}$$

Then we have

$$\lim_{m\to\infty} \lim_{k\to\infty} r\left(\tfrac{t}{k}A_m\right)^k (u_0) \overset{\text{1st Brezis–Pazy}}{=} \lim_{m\to\infty} S_m(t)(u_0)$$

$$\overset{\text{2nd Brezis–Pazy}}{=} S(t)(u_0)$$

$$\overset{\text{Crandall–Liggett}}{=} u(t)$$

Let $K \in \mathbb{N}$ with $K > \alpha t$.

This corresponds to $F_n : \mathcal{B}^{K+1} \to \mathcal{B}^{K+1}$ defined as

$$\big(F_n(v_n)\big)_k := (v_n)_k - \big(I - \tfrac{t}{K} A_m\big)^{-1}\big((v_n)_{k-1}\big)$$

for $k = 0, ..., K$ with $(v_n)_0 = u_0$

(For linear problems: Sanz-Serna & Palencia, 1985)

Question: More general formula for $r\big(\tfrac{t}{K} A_m\big)$?

Fix $t \geq 0$. Take $K \in \mathbb{N}$ with $K > \alpha t$ and $\tau = \frac{t}{K}$.

Take $\nu, \nu_i \in \mathbb{N}$, $z_0, z_{ij} \in \mathbb{C}$, $c_i \in \mathbb{R}$ with $c_i > \alpha\tau$

for all $i = 1, ..., \nu$, $j = 1, ..., \nu_i$.

For all $v \in \overline{D(A)}$, define $r(\tau A_m) : \mathcal{B} \to \mathcal{B}$ as

$$\big(r(\tau A_m)\big)(v) := z_0 v + \sum_{i=1}^{\nu} \sum_{j=1}^{\nu_i} z_{ij} \big(I - \tfrac{\tau}{c_i} A_m\big)^{-j}(v)$$

Then we have for all $v_n \in X_n = \mathcal{B}^{K+1}$ that

$$\big(F_n(v_n)\big)_k := (v_n)_k - \big(r(\tau A_m)\big)\big((v_n)_{k-1}\big)$$

Proposition

Define $\Lambda_{c_i} := \frac{1}{1-\frac{\tau}{c_i}\alpha} > 1$ *and* $C := |z_0| + \sum_{i=1}^{\nu} \sum_{j=1}^{\nu_i} |z_{ij}| \Lambda_{c_i}^{j}$

and $a_K := \dfrac{C-1}{C^K - 1}$

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Let $X_n := \left(\mathcal{B}^{K+1}, \|\cdot\|_{X_n}\right)$, $\|v_n\|_{X_n} := a_K \sum_{k=0}^{K} \left\|(v_n)_k\right\|_{\mathcal{B}}$

and $Y_n := \left(\mathcal{B}^{K+1}, \|\cdot\|_{Y_n}\right)$, $\|v_n\|_{Y_n} := \sum_{k=0}^{K} \left\|(v_n)_k\right\|_{\mathcal{B}}$

for $v_n = \left((v_n)_0, ..., (v_n)_K\right) \in X_n$, $(v_n)_k \in \mathcal{B}$

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for $v_n = \left((v_n)_0, ..., (v_n)_K\right) \in X_n, \ (v_n)_k \in \mathcal{B}$

Then F_n *is N-stable with* $S = 1$ *.*

Proof

A_m are m-dissipative of type $\omega_m \leq \alpha$

$$\Rightarrow \quad \left(I - \tfrac{\tau}{c_i} A_m\right)^{-1} \quad \text{are Lipschitz with} \quad \frac{1}{1-\frac{\tau}{c_i}\omega_m} \leq \frac{1}{1-\frac{\tau}{c_i}\alpha} = \Lambda_{c_i}$$

$$\Rightarrow \quad \|r(\tau A_m)(v) - r(\tau A_m)(z)\|_{\mathcal{B}} \leq \underbrace{\left(|z_0| + \sum_{i=1}^{\nu} \sum_{j=1}^{\nu_i} |z_{ij}| \Lambda_{c_i}^{j}\right)}_{C} \|v - z\|_{\mathcal{B}}$$

Remember:

$$(v_n)_k = \big(F_n(v_n)\big)_k + \big(r(\tau A_m)\big)\big((v_n)_{k-1}\big)$$

Proof

By induction:

$$\|(v_n)_k - (z_n)_k\|_{\mathcal{B}} \le \sum_{j=0}^{K} C^{k-j} \big\| \big(F(v_n)\big)_j - \big(F(z_n)\big)_j \big\|_{\mathcal{B}}$$

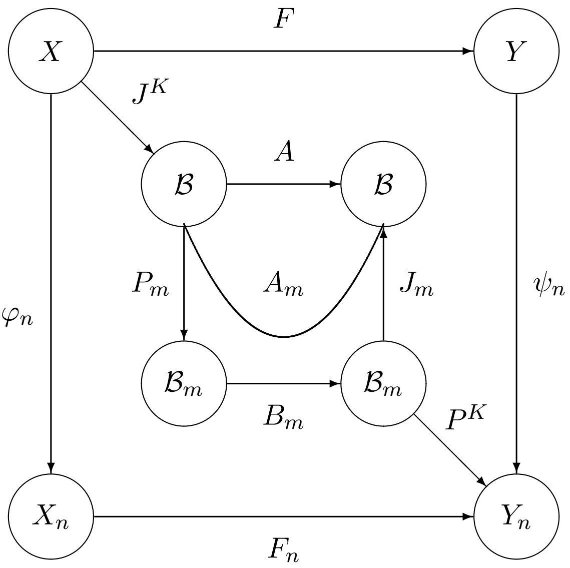
Hence:

$$\begin{aligned} \|v_n - z_n\|_{X_n} &= \frac{C-1}{C^K - 1} \sum_{k=0}^{K} \|(v_n)_k - (z_n)_k\|_{\mathcal{B}} \\ & \frac{C-1}{C^K - 1} \sum_{k=0}^{K} \sum_{j=0}^{K} C^{k-j} \big\| \big(F(v_n)\big)_j - \big(F(z_n)\big)_j \big\|_{\mathcal{B}} \\ & \frac{C-1}{C^K - 1} \sum_{k=0}^{K} \frac{C^K - 1}{C-1} \big\| \big(F(v_n)\big)_j - \big(F(z_n)\big)_j \big\|_{\mathcal{B}} \qquad \square \end{aligned}$$

Summary

- General nonlinear rational approximations are N-stable
- Question: Compatible norms? $\rightarrow$ assumption on φ_n

$$v \in X: \quad \|\varphi_n v\|_{X_n} = a_K \sum_{k=0}^{K} \|(\varphi_n v)_k\|_{\mathcal{B}} \xrightarrow{n\to\infty} \|v\|_X$$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