# Convergence of the Rothe Method Applied to Operator DAEs

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- Operator DAE (abstract DAE, PDAE)

$$\begin{aligned} \dot{u}(t) + \mathcal{K}u(t) + \mathcal{B}^*\lambda(t) &= \mathcal{F}(t), \\ \mathcal{B}u(t) \qquad\qquad\qquad &= \mathcal{G}(t) \end{aligned}$$

- Initial condition $u(0) = u_0$
- Example: (Navier-)Stokes equations

- Operator DAE (abstract DAE, PDAE)

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- Initial condition $u(0) = u_0$
- Example: (Navier-)Stokes equations
- Outline
  - what are (operator) DAEs?
  - regularization
  - implicit Euler scheme
  - convergence results for $u$ and $\lambda$

- DAE = differential-algebraic equation

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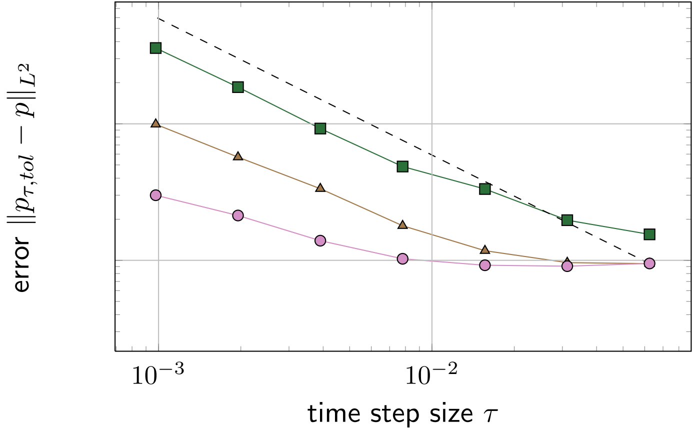
- Short introduction to DAEs:
  - derivative of $g$ appears in solution
  - sensible to perturbations
  - need consistent initial condition
  - ODE solvers may not converge

  - index: measures 'distance' to an ODE
  - semi-explicit DAE of index 2

- Semi-explicit Euler scheme applied to index-2 DAE
- Evolution of the error in the pressure variable
- Different tolerances in iterative solver



- Constrained PDE in the weak sense

$$\begin{aligned} \dot u(t) + \mathcal{K}u(t) + \mathcal{B}^*\lambda(t) &= \mathcal{F}(t) \quad \text{in } \mathcal{V}^*, \\ \mathcal{B}u(t) &= \mathcal{G}(t) \quad \text{in } \mathcal{Q}^* \end{aligned}$$

- e.g. $\mathcal{V} = [H_0^1(\Omega)]^d$, $\mathcal{Q} = L^2(\Omega)/\mathbb{R}$
- Gelfand triple $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*$

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- Gelfand triple $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*$
- $\mathcal{F} \in L^2(0,T;\mathcal{V}^*)$, $\mathcal{G} \in H^1(0,T;\mathcal{Q}^*)$
- Operators: $\mathcal{K}\colon \mathcal{V} \to \mathcal{V}^*$ (linear, positive, continuous)
- $\mathcal{B}\colon \mathcal{V} \to \mathcal{Q}^*$ (linear, right-inverse)
- Solution: $u \in L^2(0,T;\mathcal{V})$ with $\dot{u} \in L^2(0,T;\mathcal{V}^*)$
  $\lambda \in L^2(0,T;\mathcal{Q})$

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- Initial condition $u(0) = u_0 \in \mathcal{H}$
  - $u \in C([0,T], \mathcal{H})$
  - consistency:

$$u_0 \in \mathcal{V}\colon \qquad u_0 = \mathcal{B}^- \mathcal{G}(0) + a_\mathcal{V}, \; a_\mathcal{V} \in \ker \mathcal{B} \subset \mathcal{V},$$

$$u_0 \in \mathcal{H}\colon \qquad u_0 = \mathcal{B}^- \mathcal{G}(0) + a_\mathcal{H}, \; a_\mathcal{H} \in \overline{\ker \mathcal{B}}^{\mathcal{H}}$$

- Special case: $\mathcal{G} \equiv 0$
- Work on the kernel $\mathcal{V}_\mathcal{B} := \ker \mathcal{B}$

$$\dot{u}(t) + \mathcal{K}u(t) + \mathcal{B}^*\lambda(t) = \mathcal{F}(t) \quad \text{in } \mathcal{V}_\mathcal{B}^*$$

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- General case: $\mathcal{G} \not\equiv 0$
- e.g. optimal control problems constrained by a fluid flow
- Apply regularization [AH14]
- 'Index reduction' for operator equations
  - advantageous for analysis of the Rothe method
  - semi-discretized system has index 1

- Regularized operator DAE
  - split velocity $u = u_1 + u_2$, $u_1 \in \mathcal{V}_{\mathcal{B}}$
  - add derivative of constraint
  - introduce new variable $v_2 := \dot{u}_2$

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\begin{aligned}
\dot{u}_1(t) + v_2(t) + \mathcal{K}\big(u_1(t) + u_2(t)\big) + \mathcal{B}^*\lambda(t) &= \mathcal{F}(t) \quad \text{in } \mathcal{V}^*, \\
\mathcal{B}u_2(t) &= \mathcal{G}(t) \quad \text{in } \mathcal{Q}^*, \\
\mathcal{B}v_2(t) &= \dot{\mathcal{G}}(t) \quad \text{in } \mathcal{Q}^*
\end{aligned}
$$

- Differential variable: $u_1$
- 'Algebraic' variables: $u_2$, $v_2$, $\lambda$
- Initial condition $u_1(0) = a_{\mathcal{H}}$

- Implicit Euler scheme with discrete derivative $D$
- 1st equation, tested with $v \in \mathcal{V}_\mathcal{B}$

$$(Du_1^j, v) + \langle \mathcal{K} u_1^j, v\rangle = \langle \mathcal{F}^j, v\rangle - (\mathcal{B}^- \dot{\mathcal{G}}^j, v) - \langle \mathcal{K}\mathcal{B}^-\mathcal{G}^j, v\rangle$$

- $\mathcal{F}^j$ defined by integral means
- a priori estimate: use test function $v = u_1^j$

- Define global approximation

$$U_{1,\tau}(t) := \begin{cases} u_1^0 & \text{for } t = 0, \\ u_1^j & \text{for } t \in ]t_{j-1}, t_j] \end{cases}$$

- Regularization allows to apply PDE methods for differential part
  - existence of solutions
  - a priori estimates
  - define global approximations
  - boundedness
  - extract converging subsequence

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**Theorem**

*Assume* $\mathcal{F} \in L^2(0,T;\mathcal{V}^*)$, $\mathcal{G} \in H^1(0,T;\mathcal{Q}^*)$, *and* $u_1^0 = u_\mathcal{V} \in \mathcal{V}_\mathcal{B}$. *Then, the weak limit* $U_1 \in L^2(0,T;\mathcal{V}_\mathcal{B})$ *of the sequence* $\{U_{1,\tau}\}$ *solves the operator DAE in* $\mathcal{V}_\mathcal{B}^*$.

- What happens with $\lambda$?
- 1st equation, tested with $v \in \mathcal{V}_{\mathcal{B}}^{\perp}$

$$\langle \mathcal{B}^* \lambda^j, v \rangle = \langle \mathcal{F}^j, v \rangle - (D u_1^j, v) + \ldots$$

## Theorem

*Assume $\mathcal{F} \in L^2(0,T;\mathcal{V}^*)$, $\mathcal{G} \in H^1(0,T;\mathcal{Q}^*)$, and $u_1^0 = u_{\mathcal{V}} \in \mathcal{V}_{\mathcal{B}}$. Then, the sequence $\{\Lambda_\tau\}$ converges to the solution of the operator DAE only in the sense of distributions.*

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## Theorem (with additional regularity)

*Assume $\mathcal{F} \in L^2(0,T;\mathcal{H}^*)$, $\mathcal{G} \in H^1(0,T;\mathcal{Q}^*)$, and $u_1^0 = u_{\mathcal{V}} \in \mathcal{V}_{\mathcal{B}}$. Then, the weak limits $U_1$, $U_2$, $V_2$, and $\Lambda$ solve the regularized operator DAE.*

- Operator DAEs

  e.g. flow equations, Maxwell equations

  - stability issues similar as for DAEs
  - need consistent initial conditions
  - regularization (index reduction)

- Different convergence behavior for $u$ and $\lambda$

  - depends on regularity assumptions on the data

- Regularization also for second-order systems

  e.g. elastodynamics with boundary conditions (coupled systems)

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