Solutions of the exercises – Lecture C – without problem 8

Problem 1.

It is a classical result, that under the condition $L := \max\{1, \sup_{x\in\mathbb{R}} |F'(x)|\} < \infty$ the function Φ is uniquely defined on the whole space $\mathbb{R}\times\mathbb{R}$ and it is also continuous. We will use the following property of Φ: for an arbitrary $\epsilon > 0$ and $a, b \in \mathbb{R}, a < b$ there is a sufficiently small time $t_\epsilon = t_\epsilon(a,b,f) > 0$ such that

$$\Phi(t,s) \in \mathbb{R}\setminus[a-1/2, b+1/2] \text{ for all } t \in [0,t_\epsilon], s \in \mathbb{R}\setminus[a-1,b+1] \tag{1}$$

and

$$|\Phi(t,s) - s| < \epsilon \text{ for all } t \in [0,t_\epsilon], s \in [a-1,b+1]. \tag{2}$$

Fact (1): Note that from continuity of Φ we have $t_1 > 0$ such that $\Phi(t, b+1) > b+1/2$ and $\Phi(t, a-1) < a-1/2$. If we would have $t' \in [0,t_1]$ and $b' \geq b+1$ such that $\Phi(t',b') \leq b+1/2$ then becouse the function $t \in [0,t'] \to \Phi(t,b')$ is continuous and its image includes the interval $[b+1/2, b']$ we also would have $t'' \in [0,t']$ such that $\Phi(t'',b') = b+1$. Then we would arrive (using the well-known semigroup property $\Phi(t+t';s) = \Phi(t,\Phi(t',s))$) at the contradiction

$$b+1/2 \geq \Phi(t',b') = \Phi(t'-t'', \Phi(t'',b')) = \Phi(\underbrace{t'-t''}_{\in[0,t_1]}, b+1) > b+1/2.$$

Therefore $\Phi(t,s) > b+1/2$ for all $t \in [0,t_1]$ and $s \geq b+1$. Similarly we obtain $\Phi(t,s) < a-1/2$ for all $t \in [0,t_1]$ and $s \leq a-1$.

Fact (2): Integrating from 0 to t the differential equation for y in the layout of the exercises we get $y(t) - s = \int_0^t F(y(\tau))\mathrm{d}\tau$. Therefore

$$|y(t)-s| \leq \int_0^t |F(y(\tau)) - F(s)| + |F(s)|\mathrm{d}\tau \leq t|F(s)| + \int_0^t L|y(\tau)-s|\mathrm{d}\tau.$$

The usual application of Gronwall – Bellman lemma yields

$$|y(t)-s| \leq t|F(s)| + \int_0^t |F(s)|\tau L \mathrm{e}^{L(t-\tau)}\mathrm{d}\tau = \cdots = \frac{|F(s)|}{L}(\mathrm{e}^{Lt}-1). \tag{3}$$

Now we have

$$|F(s)| \leq |F(s)-F(0)| + |F(0)| \leq L|s| + F(0) \leq \max_{s\in[a-1,b+1]} L|s| + F(0) := C < \infty.$$

For $\epsilon > 0$ we can choose $t_2 > 0$ such that $\frac{C}{L}(e^{Lt}-1) < \epsilon$ for all $t \in [0,t_2]$. For these t and $s \in [a-1,b+1]$ we arrive at $|\Phi(t,s)-s| = |y(t)-s| < \epsilon$.

Therefore (1) and (2) are valid for $t_\epsilon := \min\{t_1,t_2\}$.

Part a) of the exercise: The semigroup property of $T(t)$ is a straightforward consequence of the semigroup property of Φ. $T(t)$ is well-defined, that is $T(t)f \in X$ for all $f \in X$. To see this, choose $f \in X, t > 0$ ($t = 0$ is trivial). $T(t)f$ is continuous (becouse of the continuity of Φ) so we have to show that its limits are zero at $\pm\infty$. Choose $\epsilon > 0$. For a sufficiently long intervall $[a,b]$ we have $|f(s)| < \epsilon$ for all $s \in \mathbb{R}\setminus[a,b]$. Modificating the ideas showing (1) and (2) we can establich the existence of $A \leq a$ and $B \geq b$ such that $\Phi(t,s) \in \mathbb{R}\setminus[a,b]$ for all $s \in \mathbb{R}\setminus[A,B]$. Therefore $|(T(t)f)(s)| = |f(\Phi(t,s))| < \epsilon$ for $s \in \mathbb{R}\setminus[A,B]$ and we are done.

From $\|T(t)f\| = \sup_{s\in\mathbb{R}} |f(\Phi(t,s))| = \sup_{r\in\mathbb{R}:\exists s\in\mathbb{R}:r=\Phi(t,s)} |f(r)| \le \|f\|$ we have that T is a contraction semigroup. Now we show strong continuity. Becouse of locally boundedness it is enough to prove this for functions $f \in X$ which are C^1 and has compact support. Fix such a function f and also $\epsilon > 0$. Let $[a,b]$ be an interval for which $f(s) = f'(s) = 0$ for all $s \in \mathbb{R}\setminus[a,b]$. Then for $t \in [0, t_\epsilon(a,b,f)]$ using (1) and (2) we get

$$\|T(t)f - f\| = \sup_{s\in[a-1,b+1]} |f(\Phi(s,t)) - f(s)| \le \max_{r\in[a-1,b+1]} |f'(r)|\epsilon$$

which yields strong continuity.

To identify the generator denote by N_F the set of zeros of F, that is

$$N_F := \{s \in \mathbb{R} : F(s) = 0\}.$$

Let $(A, D(A))$ be the generator. Note that the expression $\frac{f(\Phi(t,s))-f(s)}{t}$ has a uniform limit function (uniform in $s \in \mathbb{R}$) in X as $t \to 0^+$ only if it has a pointwise limit. Therefore define the candidate $(B, D(B))$ as follows

$$\begin{aligned} D(B) &:= \left\{ f \in X : f'(s) \text{ exists for } s \in \mathbb{R}\setminus N_F \text{ and } g \in X \text{ for } g(s) := \begin{cases} f'(s)F(s) & \text{if } s \in \mathbb{R}\setminus N_F \\ 0 & \text{if } s \in N_F \end{cases} \right\} \\ (Bf)(s) &:= \begin{cases} f'(s)F(s) & \text{for } s \in \mathbb{R}\setminus N_F \\ 0 & \text{for } s \in N_F \end{cases}, \quad \text{for } f \in D(B). \end{aligned}$$

From what we said before we have $A \subset B$. On the other choose $f \in D(B)$. For $s \in N_F, t > 0$ we get simply

$$\left| \frac{f(\Phi(t,s)) - f(s)}{t} - Bf(s) \right| = 0.$$

Let $s \in \mathbb{R}\setminus N_F$ then

$$\left| \frac{f(\Phi(t,s)) - f(s)}{t} - Bf(s) \right| = \left| \frac{1}{t} \int_0^t \underbrace{f'(\phi(\tau,s))F(\phi(\tau,s)) - f'(s)F(s)}_{:=w(\tau,s)} \mathrm{d}\tau \right|.$$

Choose $\epsilon > 0$. Then we have $[a,b], a < b$ such that $|f'(s)F(s)| < \epsilon/2$ for all $s \in \mathbb{R}\setminus[a,b]$. Now we apply again (1) and (2). Denote by $t_\epsilon := t_\epsilon(a,b,f'(\cdot),F(\cdot)) > 0$. For $\tau \in (0, t_\epsilon]$ we obtain $|w(\tau,s)| < \epsilon$ in the cases $s \in [a-1,b+1]N_F$ and $s \in \mathbb{R}\setminus(N_F \cup [a-1,b+1])$. These considerations implies

$$\left\| \frac{f(\Phi(t,\cdot)) - f(\cdot)}{t} - Bf(\cdot) \right\| < \epsilon$$

for $t \in (0, t_\epsilon]$ and therefore $A = B$.

Part b) of the exercise: The abstract Cauchy problem can be formuleted as follows

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = (Au(t,\cdot))(x) & \text{for } t \ge 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

We can associete with it in the case $u \in C^1(\mathbb{R}\times\mathbb{R})$ the following PDR

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - F(x)\frac{\partial u(t,x)}{\partial x} = 0 & \text{for } t \ge 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

Using the method of characteristics we are seeking for a solution of

$$t' = 1, x' = -F(x), t(0) = 0, x(0) = x_0$$

this can be easily found as $\xi_{0,x_0}(s) = (s, \Phi(-s, x_0))$ for $s \geq 0, x_0 \in \mathbb{R}$. We know that the solution u is constant along this curve, therefore

$$u(t,x) = u(t, \Phi(-t, \underbrace{\Phi(t,x)}_{:=z})) = u(\xi_{0,z}(t)) = u_0(z) = u_0(\Phi(t,x)) = (T(t)u_0)(x).$$

Problem 2.

The general Taylor formula for $n \geq 1$ is the following one for $f \in D(A^n)$

$$T(t)f = f + tAf + \frac{t^2}{2!}A^2 f + \cdots + \frac{t^{n-1}}{(n-1)!}A^{n-1}f + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}T(s)A^n f \mathrm{d}s. \tag{4}$$

This is a straightforward consequence of $\phi(t) - \phi(0) = \int_0^t \phi'(s)\mathrm{d}s$ for continuosly differentiable function $\phi : [0,t] \to X$ defined as

$$\phi(s) := T(s)f + (t-s)T(s)A(f) + \frac{(t-s)^2}{2!}T(s)A^2 f + \cdots + \frac{(t-s)^{n-1}}{(n-1)!}T(s)A^{n-1}f.$$

Indeed, it is easy to see, that $\phi(t) = T(t)f$ and $\phi(0) = (-1)$ times the right-hand side of (4) except the integral. Becouse $f \in D(A^n)$ we have that ϕ is continuosly differentiable. The last task is to find the derivative, easy calculations shows

$$\phi'(s) = \frac{(t-s)^{n-1}}{(n-1)!}T(s)A^n f.$$

Problem 3.

Georg Spielberger show me this solution. From previous problem we have

$$Af = \frac{1}{t}(T(t)f - f) - \frac{1}{t}\int_0^t (t-s)T(s)A^2 f \mathrm{d}s, \qquad f \in D(A^2), t > 0.$$

Using that T is a contraction we get

$$\|Af\| \leq \frac{2}{t}\|f\| + \frac{t}{2}\|A^2 f\|, \qquad t > 0. \tag{5}$$

In the case $\|A^2 f\| = 0$ we get $\|Af\| \leq \lim_{t\to\infty} \frac{2}{t}\|f\| = 0$ therefore $\|Af\|^2 \leq \|A^2 f\|\|f\|$ is valid. On the other hand for $\|A^2 f\| > 0$ it is enough to substitute $t := 2\sqrt{\frac{\|f\|}{\|A^2 f\|}}$ to (5) and we are done.

Problem 4.

The eqiuvalence of the norm follows from

$$|||f||| = \sup\{\|T(t)f\| : t \geq 0\} \leq \sup\{M\|f\| : t \geq 0\} = M\|f\|$$

and

$$\|f\| = \|T(0)f\| \le \sup\{\|T(t)f\| : t \ge 0\} = |||f|||.$$

In the new norm the semigruop T is a contraction becuouse of

$$|||T(t)f||| = \sup\{\|\underbrace{T(s)T(t)}_{=T(s+t)}f\| : s \ge 0\} \le \sup\{\|T(r)f\| : r \ge 0\} = |||f|||.$$

Problem 5.

We give a solution in three situations, from simplest one to the general case:

I. Assume that $\|T(t)\| \le 1$. Then

$$\left\|\left(T\left(\frac{t}{n}\right)S\left(\frac{t}{n}\right)\right)^n\right\| \le \underbrace{\left(\left\|T\left(\frac{t}{n}\right)\right\|\left\|S\left(\frac{t}{n}\right)\right\|\right)\cdots\left(\left\|T\left(\frac{t}{n}\right)\right\|\left\|S\left(\frac{t}{n}\right)\right\|\right)}_{n \text{ times}} \le \left\|S\left(\frac{t}{n}\right)\right\|^n \le \mathrm{e}^{\|B\|t}.$$

II. Assume that $\|T(t)\| \le M$ for some constant $M \ge 1$. We can apply results from Problem 4 – we use the notation $|||\cdot|||$ for the new norm in X and also for the induced operator norm on $\mathcal{L}(X)$. We know that T is a contraction in the new operator norm $|||\cdot|||$. Therefore from the previous step we get

$$\left|\left|\left|\left(T\left(\frac{t}{n}\right)S\left(\frac{t}{n}\right)\right)^n\right|\right|\right| \le \mathrm{e}^{|||B|||t}.$$

Again from Problem 4. we obtain for $f \in X$

$$\left\|\left(T\left(\frac{t}{n}\right)S\left(\frac{t}{n}\right)\right)^n f\right\| \le \left|\left|\left|\left(T\left(\frac{t}{n}\right)S\left(\frac{t}{n}\right)\right)^n f\right|\right|\right| \le \mathrm{e}^{|||B|||t}|||f||| \le M\mathrm{e}^{|||B|||t}\|f\|.$$

III. In the general situation we have $\|T(t)\| \le M\mathrm{e}^{\omega t}$. Now we use II. for a semigroup $\tilde{T}$ defined as $\tilde{T}(t) := \mathrm{e}^{-\omega t}T(t)$ (see part b) of Exercise 3 in Lecture 2) – because of $\|\tilde{T}(t)\| \le M$ the assumption of II. is fullfilled. So we get

$$\left\|\left(\tilde{T}\left(\frac{t}{n}\right)S\left(\frac{t}{n}\right)\right)^n\right\| \le M\mathrm{e}^{|||B|||t}.$$

Now returning to T we finally arrive at

$$\left\|\left(T\left(\frac{t}{n}\right)S\left(\frac{t}{n}\right)\right)^n\right\| = \left\|\left(\mathrm{e}^{\omega\frac{t}{n}}\tilde{T}\left(\frac{t}{n}\right)S\left(\frac{t}{n}\right)\right)^n\right\| \le M\mathrm{e}^{(\omega+|||B|||)t}.$$

Problem 6.

The Crank-Nicolson scheme F (cf. Example 4.4 from Lecture 4) is well-defined. To show consistency we use Proposition 4.5 from Lecture 4. We see for $f \in D(A)$ and $h \in (0, 1/\omega]$ (here $\omega > 0$ is an arbitrary number for which T is of the type (M,ω)) that

$$\begin{aligned}\frac{F(h)f - f}{h} &= \frac{[I + h/2A][I - h/2A]^{-1}f - f}{h} = \frac{[I + h/2A - I + h/2A][I - h/2A]^{-1}f}{h}\\ &= A[I - h/2A]^{-1}f = \frac{1}{h/2}R\left(\frac{1}{h/2}, A\right)Af.\end{aligned}$$

From part b) of Proposition 4.8 we have $\frac{1}{h/2}R\left(\frac{1}{h/2}, A\right)Af \to Af$ as $h \to 0^+$. Therefore $\frac{\mathrm{d}F(t)f}{\mathrm{d}t}|_{t=0} = Af$ for all $f \in D(A)$ and we are done.

Problem 7.

We show for a strongly continuous scheme $F : [0,\infty) \to \mathcal{L}(X)$ that the following three statements are equivalent:

(A) for all $t_1 > 0$ there is $M_1 = M_1(t_1) \geq 0$ such that

$$\|(F(h))^n\| \leq M_1, \qquad \text{for } h \geq 0, n \in \mathbb{N}, hn \leq t_1,$$

(B) there is $t_2 > 0$ and $M_2 \geq 0$ such that

$$\|(F(h))^n\| \leq M_2, \qquad \text{for } h \geq 0, n \in \mathbb{N}, hn \leq t_2,$$

(C) for all $t_3 \geq 0$ there is $M_3 = M_3(t_3) \geq 0$ such that

$$\left\|\left(F\left(\frac{t}{n}\right)\right)^k\right\| \leq M_3, \qquad \text{for } t \in [0,t_3], n \in \mathbb{N}, k \in \{1,2,\ldots,n\}.$$

Note that it is necessary that $M_1, M_2, M_3 \geq 1$ becouse of $\|F(0)\| = 1$. We use notation $c(t) := \sup_{s\in[0,t]} \|F(s)\|$. Again $c(t) \geq 1$ and from part a) of Propostition 2.2 (Lecture 2) we have also $c(t) < \infty$.

$(A) \Rightarrow (B)$ is trivial. We prove the reversed fact $(B) \Rightarrow (A)$. Choose $t_1 > 0$. For case $t_1 \leq t_2$ it suffices to set $M_1(t_1) := M_2$. Therefore assume that $t_1 > t_2$. If $h > t_2/2$ and $n \in N$ is such that $nh \leq t_1$ then

$$\|[F(h)]^n\| \leq C(t_1)^n.$$

From $t_2/2 < h \leq t_1/n$ we get $n < \frac{2t_1}{t_2}$ so in this case

$$\|[F(h)]^n\| \leq C(t_1)^{\frac{2t_1}{t_2}}.$$

On the other hand for $h \leq t_2/2, n \in \mathbb{N}$ let $j_h \geq 2$ denote the greatest integer for which $j_h h \leq t_2$. Let $p_h \in \mathbb{N}, q_h \in \{0,1,\cdots,j_h\}$ be the numbers uniquely determined by the equation $n = p_h j_h + q_h$. Now we get

$$\|[F(h)]^n\| = \left\|[F(h)]^{q_h}\left([F(h)]^{j_h}\right)^{p_h}\right\| \leq M_2^{p_h+1}.$$

From $nh \leq t_1$, $(j_h+1)h > t_2$ and $h \leq t_2/2$ we obtain

$$p_h \leq \frac{n}{j_h} < \frac{t_1/h}{t_2/h-1} = \frac{t_1}{t_2-h} \leq \frac{2t_1}{t_2}$$

so

$$\|[F(h)]^n\| \leq M_2^{\frac{2t_1}{t_2}+1}.$$

Therefore the following setting gives the reversed statement

$$M_1(t_1) := \begin{cases} M_2 & \text{for } t_1 \leq t_2, \\ \max\left\{C(t_1)^{\frac{2t_1}{t_2}}, M_2^{\frac{2t_1}{t_2}+1}\right\} & \text{for } t_1 > t_2. \end{cases}$$

Now we show $(A) \Leftrightarrow (C)$. For $(A) \Rightarrow (C)$ let $t_3 \geq 0$ be arbitrary. Seting $h := t/n$ we arrive at $\|F(t/n)^k\| = \|F(h)^k\| \leq M_1(t_3)$ and so $M_3(t_3) := M_1(t_3)$ is a good choice. $(C) \Rightarrow (A)$ is similarly simple. Let $t_1 > 0$ with a setting $t := hn \in (0,t_1]$ we get $\|F(h)^n\| = \|F(t/n)^n\| \leq M_3(t_1)$, therefore $M_1(t_1) := M_3(t_1)$ is a right expression for our purpose.