Lecture 9 — Solutions

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Exercise 1. For $A \in \mathcal{L}(X)$ and $z \in \mathbb{C}$ define

$$T(z) = \mathrm{e}^{zA} := \sum_{n=0}^{\infty} \frac{z^n A^n}{n!}.$$

Then T is an analytic semigroup.

Proof. For all $z \in \mathbb{C}$ we have

$$\sum_{n=0}^{\infty} \left\| \frac{z^n A^n}{n!} \right\| = \sum_{n=0}^{\infty} \frac{\big(|z| \cdot \|A\|\big)^n}{n!} = \mathrm{e}^{|z| \cdot \|A\|},$$

i.e. the series $\sum\limits_{n=0}^{\infty} \frac{z^n A^n}{n!}$ is absolutely convergent. Obviously, we have $T(0) = I$. The absolute convergence of the exponential series allows to compute the product of $T(z)$ and $T(\omega)$ for all $z, \omega \in \mathbb{C}$ via the Cauchy product formula:

$$T(z)T(\omega) = \sum_{n=0}^{\infty} \frac{z^n A^n}{n!} \sum_{m=0}^{\infty} \frac{\omega^m A^m}{m!} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{z^m A^m \omega^{n-m} A^{n-m}}{m!(n-m)!}$$

$$= \sum_{n=0}^{\infty} \frac{A^n}{n!} \sum_{m=0}^{n} \binom{n}{m} z^m \omega^{n-m} = \mathrm{e}^{(z+\omega)A} = T(z+\omega).$$

The limit

$$\lim_{h \downarrow 0} \frac{T(z+h) - T(z)}{h} x = \lim_{h \downarrow 0} \frac{T(h) - I}{h} T(z) x$$

exists for all $x \in X$ and $z \in \mathbb{C}$. That's mean that T is holomorphic. Now prove that for every $\theta' \in (0, \theta)$ the equality

$$\lim_{\substack{z \to 0 \\ z \in \sum_{\theta'}}} T(z) f = f \text{ holds for all } f \in X.$$

$$\lim_{\substack{z\to 0\\ z\in\sum_{\theta'}}} T(z)f = \lim_{\substack{z\to 0\\ z\in\sum_{\theta'}}} \mathrm{e}^{zA}f = \lim_{\substack{z\to 0\\ z\in\sum_{\theta'}}} \sum_{n=0}^{\infty}\frac{z^nA^n}{n!}f = \lim_{\substack{z\to 0\\ z\in\sum_{\theta'}}} \left(I+\sum_{n=1}^{\infty}\frac{z^nA^n}{n!}\right)f = f. \tag{1}$$

Thus, we obtain that T is an analytic semigroup.

Exercise 2. Show that T defined in Example 9.6 is a bounded analytic semigroup.

Proof. In Example 9.6 the semigroup was determined as

$$T(t) := S^{-1}M_{\mathrm{e}^{zm}}S, \qquad T : \Sigma_{\frac{\pi}{2}} \cup \{0\} \to \mathcal{L}(H),$$

where $S : H \to L^2$ is unitary operator, H is a Hilbert space and m in $(-\infty, 0]$. The operator $M_{\mathrm{e}^{zm}}$ is a multiplication operator which acts as follows: $(Mx)(t) = \mathrm{e}^{zm(t)}x(t)$, where $m(t) \leq 0$. Thus, from the structure of this operator, the semigroup $T(t)$ is a holomorphic.

Checking the properties of analytic semigroup.

1) Let $z_1, z_2 \in \Sigma_{\frac{\pi}{2}} \cup \{0\}$. Then

$$\begin{aligned} T(z_1+z_2) = S^{-1}M_{\mathrm{e}^{(z_1+z_2)m}}S = S^{-1}M_{\mathrm{e}^{z_1m}\mathrm{e}^{z_2m}}S = S^{-1}M_{\mathrm{e}^{z_1m}}M_{\mathrm{e}^{z_2m}}S \\ = S^{-1}M_{\mathrm{e}^{z_1m}}SS^{-1}M_{\mathrm{e}^{z_2m}}S = T(z_1)T(z_2). \end{aligned}$$

2) $T(0) = S^{-1}M_{\mathrm{e}^{0m}}S = S^{-1}S = I$.

3) For each $\theta' \in (0, \frac{\pi}{2})$ we have

$$\begin{aligned} \|T(z)f - f\| = \|S^{-1}M_{\mathrm{e}^{zm}}Sf - f\| = \|S^{-1}M_{\mathrm{e}^{zm}}Sf - S^{-1}Sf\| \\ = \|S^{-1}(M_{\mathrm{e}^{zm}} - 1)Sf\| \to 0, \end{aligned}$$

as $z \to 0$, holds for all $f \in L^2$. Thus, $T(t)$ is analytic semigroup.

4) Prove that $T(t)$ is a bounded. For all $\theta' \in (0, \frac{\pi}{2})$ we have

$$\|T(z)f\| = \|S^{-1}M_{\mathrm{e}^{zm}}Sf\| \leq \|S^{-1}\|\|M_{\mathrm{e}^{zm}}f\|\|S\| \leq \|M_{\mathrm{e}^{zm}}f\| \leq \mathrm{Const}\|f\|,$$

because function $m \leq 0$. Hence, $\sup_{z\in\Sigma_{\theta'}} \|T(z)\| < \infty$. Thus, $T(t)$ is a bounded analytic semigroup.

Exercise 3. Prove the assertions in Example 9.7.

Example 9.7. The shift semigroup on $\mathrm{L}^p(\mathbb{R})$ is not analytic. Or, more generally, if T is a strongly continuous group which is not continuous for the operator norm at $t = 0$, then T is not analytic.

Proof. Shift semigroup on $\mathrm{L}^p(\mathbb{R})$ is an isometric group. Its generator A is the differential operator with spectrum $\sigma(A) = \mathrm{i}\mathbb{R}$. Then T is not analytic, because the spectrum of the analytic semigroup fills a sector.

Exercise 4. Let X, Y be Banach spaces. Show that if A is a sectorial operator on X and $S : X \to Y$ is continuously invertible then SAS^{-1} is a sectorial operator on Y.

Proof. Let operator $B = SAS^{-1}$. Consider the following equations

$$R(\lambda, B) = \lambda I - B = \lambda I - SAS^{-1} = S(\lambda I - B)S^{-1} = SR(\lambda, A)S^{-1},$$

where $\lambda \in \rho(B)$. Thus, $\rho(A) = \rho(B)$. Operator A is a sectorial operator, so

$$\sup_{\lambda\in\mathbb{C}\setminus\{0\}} \|\lambda R(\lambda, A)\| < \infty.$$

$$\|\lambda R(\lambda, B)\| = \|\lambda SR(\lambda, A)S^{-1}\| \leq \|S\|\|\lambda R(\lambda, A)\|\|S^{-1}\| \leq \|\lambda R(\lambda, A)\| < \infty.$$

Thus, $\sup\limits_{\lambda\in\mathbb{C}\setminus\{0\}} \|\lambda R(\lambda, B)\| \leq \sup\limits_{\lambda\in\mathbb{C}\setminus\{0\}} \|\lambda R(\lambda, A)\| < \infty$. Hence, $B = SAS^{-1}$ is a sectorial operator.

Exercise 6. Suppose that A generates an analytic semigroup and that $B \in \mathcal{L}(X)$. Prove that $A + B$ generates an analytic semigroup.

Proof. As A generates an analytic semigroup we have that for $\theta \in (0, \frac{\pi}{2}]$ exists the sector

$$\sum_{\theta} := \big\{ z \in \mathbb{C} \setminus \{0\} : |arg(z)| < \theta \big\}.$$

Prove that the resolvent set $R(\lambda, A+B)$ is a bounded and then $A+B$ is a generates semigroup. Indeed, take $\theta_0 > \theta$ and consider the following equations

$$\lambda I - (A+B) = \big(I - B(\lambda I - A)^{-1}\big)(\lambda I - A),$$

$$\big(\lambda I - (A+B)\big)^{-1} = (\lambda I - A)^{-1}\big(I - B(\lambda I - A)^{-1}\big)^{-1}.$$

We have that

$$\big\|R(\lambda, A)\big\| \leq \frac{M}{1+|\lambda|} \text{ for all } \lambda \leq 0 \text{ and some } M \geq 0.$$

Therefore we have

$$\|R(\lambda, A+B)\| = \|R(\lambda, A)\| \| \sum_{n=0}^{\infty} \big(-B(\lambda I - A)^{-1}\big)^n \| \leq \frac{M_1}{1+|\lambda|} \text{ for some } M_1 \geq 0.$$

Thus, operator $A+B$ will be generates analytic semigroup.