Problem 1

To prove that $T(z)$ is an analytic semigroup we have to check conditions (i)–(iii) of Definition 9.1.

Observe first that as A is bounded

$$\left\|\sum_{n=0}^{\infty}\frac{z^n A^n}{n!}\right\| \leq \sum_{n=0}^{\infty}\frac{|z|^n\|A\|^n}{n!} = e^{|z|\|A\|}.$$

Thus the series is convergent and so we obtain conditions (i) and (iii). To prove (ii) we use the arguments analogous to those in Exercise 2.1, namely

$$\begin{aligned}
T(z)T(w) &= \left(\sum_{n=0}^{\infty}\frac{z^n A^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{w^n A^n}{n!}\right) = \sum_{l,k\geq 0}\frac{1}{l!k!}z^l w^k A^{l+k}\\
&= \sum_{m=0}^{\infty}\sum_{l=0}^{m}\frac{1}{l!(m-l)!}z^l w^{m-l}A^m = \sum_{m=0}^{\infty}\frac{1}{m!}\sum_{l=0}^{m}\frac{m!}{l!(m-l)!}z^l w^{m-l}A^m\\
&= \sum_{m=0}^{\infty}\frac{(z+w)^m A^m}{m!} = T(z+w)
\end{aligned}$$

The condition $T(0) = I$ is obvious. This completes the proof.

Problem 2

Let H be a Hilbert space and A a self-adjoint operator on H $(A = A^*)$. then there is L^2-space and a unitary operator $S : H \to L^2$ such that

$$SAS^{-1} : L^2 \to L^2,\ SAS^{-1} = M_m, \tag{1}$$

where M_m is a multiplication operator on L^2 by a real-valued function m. If A is negative, then $\sigma(A) \subseteq (-\infty, 0]$.

Let us consider an operator

$$T(z) := S^{-1}M_{e^{zm}}S : \sum_{\frac{\pi}{2}\cup\{0\}} \to L(H). \tag{2}$$

First of all let us prove that $T(z)$ is holomorphic. For an arbitrary $f \in H$ $T(z)f = (S^{-1}M_{e^{zm}}S)\{f\} = (S^{-1})\{e^{zm}Sf\}$. Let us put $A(z) := S^{-1}me^{zm}S$. Now our aim is to show that $A(z)$ is a derivative of $T(z)$ $\forall z \in \sum_{\frac{\pi}{2}\cup\{0\}}$:

(3)

$$||\frac{T(z+h)f - T(z)f}{h} - A(z)f|| \leq ||S^{-1}||||\frac{e^{(z+h)m} - e^{zm}}{h} - me^{zm}||_{L^2}||S|| \to 0,\ h \to 0$$

I.e.,

$$||\frac{T(z+h) - T(z)}{h} - A(z)|| \to 0,\ h \to 0. \tag{4}$$

The fact, that $m \leq 0$ and $z \in \sum_{\frac{\pi}{2} \cup \{0\}}$ guarantees that the corresponding integrals in L^2 are convergent, because $Re(z)m \leq 0$. The continuity of $A(z)$ can also be easily checked.

The semigroup property also holds:

$$T(z+w) = S^{-1} M_{e^{(z+w)m}} S = S^{-1} M_{e^{zm}} S S^{-1} M_{e^{wm}} S = T(z)T(w). \tag{5}$$

Now it remains to prove the equality:

$$\lim_{\substack{z \to 0 \\ z \in \sum_{\frac{\pi}{2} \cup \{0\}}}} T(z)f = f. \tag{6}$$

It holds, because

$$||T(z)f - f|| \leq ||S^{-1}|| \int_X |e^{mz} f - f|^2 d\mu ||S|| \to 0, z \to 0 \tag{7}$$

where μ is a measure in L^2.

The analytic semigroup $T(z)$ is bounded, because

$$||T(z)f|| \leq ||S^{-1}|| ||e^{zm}|| ||S|| ||f|| \leq ||S|| ||S^{-1}|| ||f||.$$

Thus, $T(z)$ is a bounded analytic semigroup.

Problem 3

Suppose that T is an analytic semigroup of angle θ. Clearly, $T : \Sigma_\theta \to \mathcal{L}(X)$ is continuous. From the definition of an analytic semigroup ((iii)) it follows that T may be extended by continuity to $\widehat{T} : \Sigma_\theta \cup \{0\} \to \mathcal{L}(X)$ and that

$$\widehat{T}(0) = I = T(0).$$

Thus T is continuous for the operator norm at $t = 0$. This contradiction completes the proof.

Problem 4

Note firstly that

$$(\lambda - SAS^{-1}) = S(\lambda - A)S^{-1}.$$

Thus if $\lambda - A$ is invertible then $\lambda - SAS^{-1}$ is also invertible and its inverse is

$$(\lambda - SAS^{-1})^{-1} = S(\lambda - A)^{-1}S^{-1}.$$

Moreover,

$$\begin{aligned} \|(\lambda - SAS^{-1})^{-1}\| &= \|S(\lambda - A)^{-1}S^{-1}\| \\ &\leq \|S\| \|(\lambda - A)^{-1}\| \|S^{-1}\| = \|(\lambda - A)^{-1}\|. \end{aligned}$$

Thus we conclude that if λ belongs to the resolvent set $\rho(A)$ then it also belongs to $\rho(SAS^{-1})$. And therefore if $\Sigma_{\frac{\pi}{2}+\delta}$ is contained in $\rho(A)$ it is also contained in $\rho(SAS^{-1})$.

Observe also that

$$\begin{aligned}\|\lambda R(\lambda, SAS^{-1})\| &= \|\lambda(\lambda - SAS^{-1})^{-1}\| = \|\lambda S(\lambda - A)^{-1}S^{-1}\| \\ &\le \|S\|\|\lambda(\lambda - A)^{-1}\|\|S^{-1}\| = \|\lambda R(\lambda, A)\|.\end{aligned}$$

Using this we obtain that if $\sup\limits_{\lambda\in\Sigma_{\frac{\pi}{2}+\delta'}} \|\lambda R(\lambda, A)\| < \infty$ then

$$\sup_{\lambda\in\Sigma_{\frac{\pi}{2}+\delta'}} \|\lambda R(\lambda, SAS^{-1})\| < \infty$$

for $\delta' \in (0, \delta)$.

The above arguments imply that if A is a sectorial operator on X and $S : X \to Y$ is continuously invertible then SAS^{-1} is a sectorial operator on Y.

Problem 5

Let us prove that (i) implies (ii). We have to consider the analytic semigroup $T(z)$ of angle θ ($z \in \sum_\theta$) with its generator A and to prove that for an arbitrary number $\omega > 0$ $\quad \tilde{T}(z) := T(z)e^{-wz} = e^{-wz}T(z)$, $z \in \sum_\theta$ is an analytic semigroup with the generator $A - w$. Let us check this:

$$\begin{aligned}&\left|\frac{\tilde{T}(h)f - \tilde{T}(0)f}{h} - Af + wf\right| = \left|\frac{T(h)e^{-wh}f - f}{h} - Af + wf\right| = \\ &= \left|e^{-wh}\frac{T(h)f - f}{h} - e^{-wh}Af + \frac{e^{-wh}f - f}{h} + wf - Af + e^{-wh}Af\right| \le \end{aligned}$$

(8)

$$\le \left|e^{-wh}\right|\left|\frac{T(h)f - f}{h} - Af\right| + \left|\frac{e^{-wh}f - f}{h} + wf\right| + \left|e^{-wh}Af - Af\right| \to 0,\ \ h \to 0$$

The inverse implication can analogously be obtained.

Now we have to prove that (i) $\iff$ (iii). It can be obtained by using Proposition 9.3 and proving analogues of Propositions 9.8, 9.18.

Problem 6

Suppose A generates an analytic semigroup T of angle θ, that $B \in \mathcal{L}(X)$. We define an analytic semigroup e^{zB} (see Example 9.4.). Then $T(z)\mathrm{e}^{zB}$ is an analytic semigroup of angle θ with generator $A+B$ (see Lecture 5, Problem 5.).

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