ISEM 2011
Solutions for the exercises of Lecture 9
Teams of Bern and Bogotá

**Exercise 1.** For $A \in \mathcal{L}(X)$ and $z \in \mathbb{C}$ define

$$T(z) = \mathrm{e}^{zA} := \sum_{n=0}^{\infty} \frac{z^n A^n}{n!}.$$

Show that T is an analytic semigroup.

S o l u t i o n . (Bogotá)

(1) $T(z)$ is a bounded linear operator for every $z \in \mathbb{C}$.
In fact, for all $z \in \mathbb{C}$, $\|\frac{z^n A^n}{n!}\| \leq \dfrac{|z|^n \|A\|^n}{n!}$, so the series $\sum\limits_{n=0}^{\infty} \frac{z^n A^n}{n!}$ converges absolutely in $\mathcal{L}(X)$. Therefore $T(z) = \sum\limits_{n=0}^{\infty} \frac{z^n A^n}{n!} \in \mathcal{L}(X)$ and we have the estimate

$$\|T(z)\| \leq \sum_{n=0}^{\infty} \left\| \frac{z^n A^n}{n!} \right\| \leq \sum_{n=0}^{\infty} \frac{|z|^n \|A\|^n}{n!} = \mathrm{e}^{|z|\|A\|}.$$

(2) $T(z)$ is satisfies the semigroup property.
Indeed, since all sums involved are absolutely convergent, we find for all $z, w \in \mathbb{C}$ that

$$\begin{aligned} T(z+w) &= \sum_{n=0}^{\infty} \frac{(z+w)^n A^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^k}{k!} A^k \frac{w^{n-k}}{(n-k)!} A^{n-k} \\ &= \sum_{n=0}^{\infty} \frac{z^n}{n!} A^n \sum_{n=0}^{\infty} \frac{w^n}{n!} A^n = T(z)T(w). \end{aligned}$$

(3) $T(\cdot)$ is holomorphic.
Since $T$ is additive, it is sufficient to prove that $T(\cdot)$ is holomorphic in 0.

$$\begin{aligned} \left\| \frac{T(h) - I}{h} - A \right\| &= \left\| \frac{1}{h} \Big( -I + \sum_{n=0}^{\infty} \frac{h^n A^n}{n!} \Big) - A \right\| = \left\| \frac{1}{h} \sum_{n=2}^{\infty} \frac{h^n A^n}{n!} \right\| \\ &= \left\| h \sum_{n=0}^{\infty} \frac{h^n A^{n+2}}{(n+2)!} \right\| \\ &\leq |h| \, \|A\|^2 \sum_{n=0}^{\infty} \frac{|h|^n}{(n+2)!} \|A\|^n \\ &\leq |h| \, \|A\|^2 \mathrm{e}^{|h|\|A\|}. \end{aligned}$$

Thus $\lim\limits_{h \to 0} \frac{T(h)-I}{h} = A$.

(4) For any $\theta \in (0, \pi/2)$ and $\theta' \in (0, \theta)$ we have that $\lim_{z\to 0} T(z)f = f$ for $z \in \Sigma_{\theta'}$.

In fact, this is an immediate consequence of the holomorphy of $T$.

**Exercise 2.** Let $H$ be a Hilbert space and $A$ be a negative self-adjoint operator on $H$. If $S : H \to L^2$ is a unitary operator such that $SAS^{-1} : L^2 \to L^2$, $SAS^{-1} = M_m$, where $M_m$ is a multiplication operator on $L^2$ by a real-valued function $m$ (with maximal domain), prove that $T(z) := S^{-1}M_{\mathrm{e}^{zm}}S$ defines a bounded analytic semigroup $T : \Sigma_{\frac{\pi}{2}} \cup \{0\} \to \mathcal{L}(H)$ generated by $A$.

S o l u t i o n. (Bern) Consider the operator $M_m$ with its maximal domain $D(M_m) = \{f \in L^2 \ : \ mf \in L^2\}$. It is easy to see that $D(M_m)$ is dense and the spectrum of $M_m$ is the essential range of $m$, see [EN00, Prop. I.4.10]. Thus the sector $\Sigma_{\frac{\pi}{2}+\frac{\pi}{2}}$ is contained in the resolvent set of $M_m$. Moreover for any $\lambda \in \Sigma_{\frac{\pi}{2}+\frac{\pi}{2}}$ we have $R(\lambda, M_m)f = \frac{1}{\lambda - m(\cdot)}f$. Thus for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$ we obtain

$$\|\lambda R(\lambda, M_m)f\|^2 = \int \underbrace{\left|\frac{\lambda}{\lambda - m(x)}\right|^2}_{\leq 1} |f(x)|^2 \, dx \leq \int |f(x)|^2 \, dx = \|f\|^2.$$

So

$$\sup_{\operatorname{Re}\lambda>0} = \|\lambda R(\lambda, M_m)\| \leq 1 < \infty.$$

Using Corollary 9.16 we conclude that $M_m$ generates a bounded analytic semigroup $\tilde{T} : \Sigma_{\frac{\pi}{2}} \cup \{0\} \to \mathcal{L}(L^2)$.

In addition, we know that $M_m$ generates the strongly continuous semigroup

$$\tilde{T}_m : [0, \infty) \to \mathcal{L}(L^2), \quad \tilde{T}_m(t)f = e^{tm}f,$$

and for any $\alpha \in (-\pi/2, \pi/2)$ the multiplication operator $M_{e^{i\alpha}m}$ with $D(M_{e^{i\alpha}m}) = D(M_m)$ generates the strongly continuous semigroup

$$\tilde{T}_{\alpha,m} : [0, \infty) \to \mathcal{L}(L^2), \quad \tilde{T}_{\alpha,m}(t)f = e^{te^{i\alpha}m}f,$$

see [EN00, Lemma II.2.9].

Now let $z \in \{z \in \mathbb{C} \ : \ \operatorname{Re} z > 0\}$. Then there exist $\alpha \in (-\pi/2, \pi/2)$ and $t > 0$ such that $z = te^{i\alpha}$. Using Proposition 9.3 we obtain

$$\tilde{T}(z)f = \tilde{T}(e^{i\alpha}t)f \overset{\text{Prop. 9.3}}{=} \tilde{T}_{\alpha,m}(t)f = e^{te^{i\alpha}m}f = e^{zm}f.$$

The similarity transformations preserves the properties of the semigroup and thus also

$$T : \Sigma_{\frac{\pi}{2}\cup\{0\}} \to \mathcal{L}(H), \quad T(z) = S^{-1}M_{e^{zm}}S$$

is a bounded analytic semigroup.

**Exercise 3.** Show that the shift semigroup on $L^p(\mathbb{R})$ is not analytic. Or, more generally, if $T$ is a strongly continuous group which is not continuous for the operator norm at $t = 0$, then $T$ is not analytic.

S o l u t i o n. (Bern) First of all we prove that if $T : \mathbb{R} \to \mathcal{L}(X)$ is an analytic group, then it must be continuous for the operator norm at $t = 0$. Let $A$ be the generator of $T$. Then by Proposition 9.20 (ii), for some $\omega > 0$ the operator $A_1 = A - \omega I$ generates a bounded analytic semigroup $T_1$. Then Proposition 9.17 implies that

$$\operatorname{ran} T_1(t) \subset D(A_1) \quad \text{for all} \quad t > 0. \tag{1}$$

On the other hand the operator $-A+\omega I$ generates a semigroup $T_2$ and for all $t \geq 0$ we have $\|T_1(t)\| \leq M$, and $\|T_2(t)\| = \|e^{\omega t}T(-t)\| \leq M$. Hence $T_1$ is a group (for more details see [EN01, Page 72]). Since $T_1(-t)$ is a semigroup, by Proposition 2.8 we obtain that $T_1(-t)D(A_1) \subseteq D(A_1)$, and therefore from (1) and the group property of $T_1$ we obtain

$$X = T_1(0)X = T_1(-t)T_1(t)X \subseteq D(A_1),$$

and consequently, $D(A_1) = X$, i.e. $A_1$ is bounded. Now Example 9.4 and the uniqueness theorem (Theorem 2.11) imply that

$$T_1(z) \equiv \sum_{n=0}^{\infty} \frac{z^n A_1^n}{n!}.$$

Since

$$\begin{aligned} 0 \leq \lim_{z\to 0} \|\sum_{n=0}^{\infty} \frac{z^n A_1^n}{n!} - I\| &= \lim_{z\to 0} \|\sum_{n=1}^{\infty} \frac{z^n A_1^n}{n!}\| \\ &\leq \lim_{z\to 0} \sum_{n=1}^{\infty} \frac{|z|^n \|A_1\|^n}{n!} \\ &= \lim_{z\to 0}(e^{|z|\|A_1\|} - 1) = 0, \end{aligned}$$

it follows that $T_1$ (hence $T$) is continuous for the operator norm at $t = 0$.

The shift semigroup on $L^p(\mathbb{R})$ with $1 \leq p < \infty$ is not analytic since it is a strongly continuous group which is not continuous for the operator norm at $t = 0$. Note that the shift semigroup is not even strongly continuous in the case $p = \infty$.

**Exercise 4.** Let $X, Y$ be Banach spaces. Show that if $A$ is a sectorial operator on $X$ and $S : X \to Y$ is continuously invertible, then $SAS^{-1}$ is a sectorial operator on $Y$.

S o l u t i o n. (Bern) Since $A$ is sectorial, there exists $\delta \in (0, \frac{\pi}{2})$ such that the sector

$$\Sigma_{\frac{\pi}{2}+\delta} = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg\lambda| < \frac{\pi}{2} + \delta\}$$

is contained in the resolvent set $\rho(A)$ of $A$, and by Remark 9.11 for every $\delta' \in (0,\delta)$ there exists $M_{\delta'} \geq 1$ such that for all $\lambda \in \overline{\Sigma}_{\frac{\pi}{2}+\delta'} \setminus \{0\}$ the following estimate holds

$$\|R(\lambda, A)\| \leq \frac{M_{\delta'}}{|\lambda|}. \tag{2}$$

On the other hand, clearly $S(\lambda I - A)S^{-1} = \lambda I - SAS^{-1}$ and thus $\rho(SAS^{-1}) = \rho(A)$, with

$$(\lambda I - SAS^{-1})^{-1} = S(\lambda I - A)^{-1}S^{-1} \quad \text{for} \quad \lambda \in \rho(A). \tag{3}$$

Now from (2) and (3) it follows that

$$\sup_{\lambda \in \Sigma_{\frac{\pi}{2}+\delta'}} \|\lambda R(\lambda, SAS^{-1})\| \leq \|S\|\|S^{-1}\| \sup_{\lambda \in \Sigma_{\frac{\pi}{2}+\delta'}} \|\lambda R(\lambda, A)\| < \infty$$

for every $\delta' \in (0,\delta)$. Therefore $SAS^{-1}$ is a sectorial operator of angle $\delta$ on $Y$.

**Exercise 5.** For a densely defined linear operator $A$ on the Banach space $X$ the following assertions are equivalent:
$(i)$ The operator A generates an analytic semigroup (of some angle).
$(ii)$ For some $\omega > 0$ the operator $A - \omega$ generates a bounded analytic semigroup (of some angle).
$(iii)$ There is $r > 0$ such that

$$\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0, |\lambda| > r\} \subset \rho(A)$$

and
$$\sup_{\substack{\mathrm{Re}\lambda > 0 \\ |\lambda| > r}} \|\lambda R(\lambda, A)\| < \infty.$$

S o l u t i o n . (Bogotá) We show that $(i) \iff (ii)$ and $(ii) \iff (iii)$.

$(i) \Rightarrow (ii)$ : Let $T(t)$ an analytic semigroup of angle $\theta$ generated by $A$, then by proposition 9.3(b) for all $\theta' \in (0,\theta)$ there exist $\omega > 0$ and $M \geq 1$ such that

$$\|T(z)\| \leq M\mathrm{e}^{\omega \mathrm{Re} z} \quad \text{for all } z \in \Sigma_{\theta'}.$$

If we define $S(z) := \mathrm{e}^{-\omega z}T(z)$ then $S(z)$ is a bounded analytic semigroup of angle $\theta'$ whose generator is $A - \omega$. In fact, that $S(z)$ is an analytic semigroup of angle $\theta'$ is inherited from $T(z)$. Since

$$\|S(z)\| = \mathrm{e}^{-\omega \mathrm{Re} z}\|T(z)\| \leq M,$$

$S(z)$ is bounded in $\Sigma_{\theta'}$. And if $B$ denote the generator of $S(z)$ then for $\lambda$ sufficiently large we have that

$$R(\lambda, B) = \int_0^\infty \mathrm{e}^{-\lambda t} S(t) \mathrm{d}t = \int_0^\infty \mathrm{e}^{-(\lambda+\omega)t} T(t) \mathrm{d}t$$
$$= R(\lambda + \omega, A) = R(\lambda, A - \omega).$$

Thus $B = A - \omega$.

$(ii) \Rightarrow (i)$ : Let us suppose that $A - \omega$ is the generator of a bounded analytic semigroup $T(z)$ of angle $\theta$. Let us define $S(z) = \mathrm{e}^{\omega z} T(z)$. It is clear that $S(z)$ satisfies all the properties of an analytic semigroup in $\Sigma_\theta$. If $B$ denotes its generator then for $\lambda$ sufficiently large we have that

$$R(\lambda + \omega, A) = R(\lambda, A - \omega) = \int_0^\infty \mathrm{e}^{-\lambda t} T(t) \mathrm{d}t$$
$$= \int_0^\infty \mathrm{e}^{-(\lambda+\omega)t} S(t) \mathrm{d}t = R(\lambda + \omega, B).$$

Thus $A = B$, i.e, $A$ is the generator of an analytic semigroup $S(z)$.

$(ii) \Rightarrow (iii)$ : Suppose that for some $\omega > 0$ the operator $A - \omega$ generates a bounded analytic semigroup of an angle $\delta \in (0, \frac{\pi}{2}]$. By Corollary 9.16 this means that the operator $A - \omega$ is sectorial of angle $\delta$. Therefore we have

$$\sup_{\lambda \in \Sigma_{\frac{\pi}{2}+\frac{\delta}{2}}} \|\lambda R(\lambda, A - \omega)\| < +\infty, \tag{4}$$

and

$$\Sigma_{\frac{\pi}{2}+\delta} \subset \rho(A - \omega). \tag{5}$$

Let $r := \max(2\omega, \omega \cot \frac{\delta}{2})$. Then from (5) it easily follows that

$$\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0, |\lambda| > r\} \subset \rho(A). \tag{6}$$

Now choose $M \in \mathbb{R}$ so that $\|\lambda R(\lambda, A - \omega)\| \leq M$ holds for all $\lambda \in \Sigma_{\frac{\pi}{2}+\frac{\delta}{2}}$. Then for every $\lambda \in w + \Sigma_{\frac{\pi}{2}+\frac{\delta}{2}}$ we have

$$\|\lambda R(\lambda, A)\| \leq \|(\lambda - \omega) R(\lambda - \omega, A - \omega)\| + \|\omega R(\lambda, A)\|$$
$$\leq M + \|\omega R(\lambda, A)\|$$
$$= M + \frac{\omega}{|\lambda|} \|\lambda R(\lambda, A)\|.$$

Thus

$$\|\lambda R(\lambda, A)\| \leq \frac{|\lambda|}{|\lambda| - \omega} M, \quad \text{whenever} |\lambda| > \omega \text{ and } \lambda \in w + \Sigma_{\frac{\pi}{2}+\frac{\delta}{2}}. \tag{7}$$

Note that

$$\{\lambda\in\mathbb{C}:\mathrm{Re}\lambda>0,|\lambda|>r\}\subset\left\{\lambda\in\mathbb{C}:|\lambda|>r,\lambda\in\omega+\Sigma_{\frac{\pi}{2}+\frac{\delta}{2}}\right\}. \quad (8)$$

Therefore

$$\sup_{\substack{\mathrm{Re}\lambda>0\\|\lambda|>r}}\|\lambda R(\lambda,A)\|\le\sup_{\substack{\mathrm{Re}\lambda>0\\|\lambda|>r}}\frac{|\lambda|}{|\lambda|-\omega}M<2M. \quad (9)$$

Now (6) together with (9) imply $(iii)$.

$(iii)\Rightarrow(ii)$ : Let $r>0$ be such that

$$\{\lambda\in\mathbb{C}:\mathrm{Re}\lambda>0,|\lambda|>r\}\subset\rho(A)$$

and

$$\sup_{\substack{\mathrm{Re}\lambda>0\\|\lambda|>r}}\|\lambda R(\lambda,A)\|<\infty.$$

Then there exists $M>0$ such that $\|\lambda R(\lambda,A)\|\le M$ for every $\lambda\in\mathbb{C}$ with $\mathrm{Re}\lambda>0$ and $|\lambda|>r$. Hence for $\lambda\in\mathbb{C}$ with $\mathrm{Re}\lambda>0$ we have that

$$\|(\lambda+r)R(\lambda+r,A)\|\le M, \quad (10)$$

and since $|\lambda+r|>|\lambda|$, (10) implies

$$\begin{aligned}\|\lambda R(\lambda,A-r)\|&\le\left|\frac{\lambda+r}{\lambda}\right|\|\lambda R(\lambda,A-r)\|=\|(\lambda+r)R(\lambda,A-r)\|\\&=\|(\lambda+r)R(\lambda+r,A)\|\le M.\end{aligned}$$

Thus

$$\{\lambda\in\mathbb{C}:\mathrm{Re}\lambda>0\}\subset\rho(A-r)\quad\text{and}\quad\sup_{\mathrm{Re}\lambda>0}\|\lambda R(\lambda,A-r)\|<\infty.$$

By Proposition 9.18, the last two relations imply that $A-r$ generates a bounded analytic semigroup.

**Exercise 6.** Suppose that $A$ generates an analytic semigroup and that $B\in\mathcal{L}(X)$. Prove that $A+B$ generates an analytic semigroup.

S o l u t i o n. (Bern) Let $T$ be a semigroup generated by $A$. By Proposition 9.3 we have $\|T(z)\|\le Me^{\omega\,\mathrm{Re}(z)}$, and the semigroup $e^{-\omega z}T(z)$ is uniformly bounded with generator $A-\omega I$. Clearly, if $A+B$ generates an analytic semigroup, then so does $A+B-\omega I$. Therefore without loss of generality we may suppose that $T$ is uniformly bounded. Then by Proposition 9.8 there exists $\theta\in(0,\frac{\pi}{2})$ such that the sector $\Sigma_{\frac{\pi}{2}+\theta}$ is contained in $\rho(A)$, and for all $\theta'\in(0,\theta)$ there exists $M_{\theta'}\ge1$ such that the estimate

$$\|R(\lambda,A)\|\le\frac{M_{\theta'}}{|\lambda|}$$

holds for all $\lambda\in\Sigma_{\frac{\pi}{2}+\theta'}$. Since $B\in\mathcal{L}(X)$

$$\|BR(\lambda,A)f\|\le\|B\|\|R(\lambda,A)f\|\le\frac{\|B\|M_{\theta'}}{|\lambda|}\|f\|,$$

it follows that for $|\lambda| > 2\|B\|M_{\theta'}$ and $\lambda \in \Sigma_{\frac{\pi}{2}+\theta'}$ we have

$$\|BR(\lambda, A)\| < \frac{1}{2},$$

and thus $I - BR(\lambda, A)$ is invertible with

$$\|\big(I - BR(\lambda, A)\big)^{-1}\| \leq \sum_{k=0}^{\infty} \|BR(\lambda, A)\|^k < \sum_{k=0}^{\infty} \frac{1}{2^k} = 2. \tag{11}$$

Then the identity

$$\begin{aligned} R(\lambda, A + B) &= \Big(\big(I - B(\lambda I - A)^{-1}\big)(\lambda I - A)\Big)^{-1} \\ &= R(\lambda, A)\big(I - BR(\lambda, A)\big)^{-1} \end{aligned}$$

together with (11) imply that for $|\lambda| > 2\|B\|M_{\theta'}$ and $\lambda \in \Sigma_{\frac{\pi}{2}+\theta'}$ the following estimate holds

$$\|R(\lambda, A + B)\| \leq \frac{2M_{\theta'}}{|\lambda|}.$$

Now taking $r := 2\|B\|M_{\theta'}$ in Proposition 9.20, we conclude that the operator $A + B$ generates an analytic semigroup.

## References

[EN00] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer, 2000.

[EN01] K.-J. Engel and R. Nagel, *A Short Course on Operator Semigroups*, Springer, 2006.