Lecture 8 — Solutions

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Exercise 1. Suppose that A generates a semigroup of type (M, ω) with $\omega < 0$ and let $\alpha \in (0,1)$. Then we have

$$X_\alpha = \left\{ f \in X : \lim_{\lambda\to\infty} \|\lambda^\alpha AR(\lambda, A)f\| = 0 \right\}.$$

Proof. Since $\big\|\lambda R(\lambda, A)\big\| \leq M$ for all $\lambda > 0$, it's clear that $\lim\limits_{\lambda\to\infty} \|\lambda^\alpha AR(\lambda, A)f\| = 0$ for $f \in D(A)$. By Proposition 8.10 we obtain

$$X_\alpha = \overline{D(A)}^{F_\alpha} \subseteq \left\{ f \in X : \lim_{\lambda\to\infty} \|\lambda^\alpha AR(\lambda, A)f\| = 0 \right\},$$

because the set on the right hand side is closed by the Favard norm. Consider $f \in X$ with $\lim\limits_{\lambda\to\infty} \|\lambda^\alpha AR(\lambda, A)f\| = 0$. By (8.4) we have

$$\|\tfrac{1}{t^\alpha}(T(t)f - f)\| \leq (t\lambda)^{1-\alpha} M \|\lambda^\alpha AR(\lambda, A)f\| + 2M(t\lambda)^{-\alpha}\|\lambda^\alpha AR(\lambda, A)f\|.$$

For $\lambda = \frac{1}{t}$ and $t \to 0$, we obtain the assertion.

Exercise 2. Let T be a semigroup of type (M, ω) with $\omega < 0$ and let $\alpha \in (0,1)$. Then X_α is a Banach space and it is invariant under the semigroup T. For all $t \geq 0$ we have $T(t) \in \mathcal{L}(X_\alpha)$.

Proof. First we show that X_α is complete. For this we take the Cauchy sequence $(f_m) \subset X_\alpha$, which is a Cauchy sequence by Lemma 8.3 in X, hence it has a limit $f \in X$. For the fixed $t > 0$ we have that

$$\big\|\tfrac{1}{t^\alpha}(T(t)f - f)\big\| = \lim_{m\to\infty} \big\|\tfrac{1}{t^\alpha}(T(t)f_m - f_m)\big\| \leq \limsup_{m\to\infty} \|f_m\|_{X_\alpha},$$

which implies that

$$\|f\|_{X_\alpha} \leq \limsup_{m\to\infty} \|f_m\|_{X_\alpha}.$$

Since $f_m - f_n \to f - f_n$ in X as $m \to \infty$, the argumentation above yields

$$\|f - f_n\|_{X_\alpha} \leq \limsup_{m\to\infty} \|f_m - f_n\|_{X_\alpha}.$$

From this we can conclude that for every $\varepsilon > 0$

$$\|f - f_n\|_{X_\alpha} \leq \varepsilon$$

holds for $n \in \mathbb{N}$ sufficiently large. This shows that $f_n \to f$ in X_α. Now we show that $f \in X_\alpha$. To this end, we fix $t > 0$. We have that

$$\begin{aligned}\left\|\frac{1}{t^\alpha}\big(T(t)f - f\big)\right\| &= \left\|\frac{1}{t^\alpha}\big(T(t)(f - f_m) - (f - f_m)\big) + \frac{1}{t^\alpha}\big(T(t)f_m - f_m\big)\right\| \\ &\leq \left\|\frac{1}{t^\alpha}(T(t)(f - f_m) - (f - f_m))\right\| + \left\|\frac{1}{t^\alpha}(T(t)f_m - f_m)\right\|.\end{aligned}$$

From this we can conclude that for every $\varepsilon > 0$

$$\|\tfrac{1}{t^\alpha}(T(t)f - f)\| \leq \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon$$

holds for $n \in \mathbb{N}$ sufficiently large. This shows that $f \in X_\alpha$. Thus, X_α is complete. The space X_α is clearly invariant under the semigroup, and we have the estimate

$$\begin{aligned}\|T(s)f\|_{X_\alpha} = \sup_{t>0} &\left\|\frac{1}{t^\alpha}(T(t)T(s)f - T(s)f)\right\| \\ &= \sup_{t>0}\left\|T(s)\frac{1}{t^\alpha}(T(t)f - f)\right\| \leq \|T(s)\| \cdot \|f\|_{X_\alpha}\end{aligned}$$

which shows that $T(s) \in \mathcal{L}(X_\alpha)$ and $\|T(s)\|_{X_\alpha} \leq \|T(s)\|$.

Exercise 3. Prove Lemma 8.3.

Lemma 8.3. *a) For $\alpha, \beta \in (0,1)$ with $\alpha < \beta$ the embeddings*

$$D(A) \hookrightarrow F_\beta \hookrightarrow F_\alpha \hookrightarrow X$$

are continuous, where $D(A)$ is equipped with the graph norm (in this case equivalent to $f \mapsto \|Af\|$).

b) The Favard norm $\|\cdot\|_{F_1}$ and the graph norm $\|\cdot\|_1$ are equivalent on $D(A)$.

Proof. b) By Proposition 8.6. we have that norms $|||f|||_{F_1}$ and $\|f\|_{F_1}$ are equivalent. Then it's sufficient to show the equivalence of the norms $\|f\|_1$ and $|||f|||_{F_1}$. Since $f \in D(A)$, it's possible to write $AR(\lambda, A)f = R(\lambda, A)Af$. Then

$$\begin{aligned}\sup_{\lambda>0} \big\|\lambda AR(\lambda, A)f\big\| &= \sup_{\lambda>0} \big\|\lambda R(\lambda, A)Af\big\| = \|Af\| \cdot \sup_{\lambda>0} \big\|\lambda R(\lambda, A)\big\| \\ &\leqslant M_0\|Af\| \leqslant M_0\big(\|Af\| + \|f\|\big) = M_0\|f\|_1.\end{aligned}$$

The operator A is invertible, then

$$\begin{aligned}\frac{\|f\| + \|Af\|}{\|A^{-1}\| + 1} &= \|Af\| = \frac{1}{\lambda}\|\lambda AR(\lambda, A)(\lambda I - A)f\| \\ &= \|\lambda AR(\lambda, A)f - AR(\lambda, A)Af\| \leqslant \|\lambda AR(\lambda, A)f\| + \|AR(\lambda, A)Af\| \\ &\leqslant \sup_{\lambda>0} \|\lambda AR(\lambda, A)f\| + \|AR(\lambda, A)Af\| = |||f|||_{F_1} + \frac{1}{\lambda}\|\lambda AR(\lambda, A)Af\| \\ &\leqslant |||f|||_{F_1} + \sup_{\lambda>0} \frac{1}{\lambda}\|\lambda AR(\lambda, A)Af\|,\end{aligned}$$

where $\frac{1}{\lambda}\|\lambda AR(\lambda, A)f\| \to 0$ if $\lambda \to \infty$.

a) The embeddings are continuous, so we have to show that

$$\|f\|_{D(A)} \geqslant \text{Const}\|f\|_{F_\beta} \geqslant \text{Const}\|f\|_{F_\alpha} \geqslant \text{Const}\|f\|_X.$$

The first inequality follows from b). Next, $\|f\|_{F_\beta} \geqslant \text{Const}\|f\|_{F_\alpha}$ because

$$\|f\|_{F_\alpha} = \sup_{t>0}\left\|\frac{1}{t^\alpha}\big(T(t)f - f\big)\right\| = \sup_{t>0}\left\|t^{\beta-\alpha}\frac{1}{t^\beta}\big(T(t)f - f\big)\right\|$$
$$\leqslant \sup_{t>0}\|t^{\beta-\alpha}\| \cdot \sup_{t>0}\left\|\frac{1}{t^\beta}\big(T(t)f - f\big)\right\| = \sup_{t>0}\left\|\frac{1}{t^\beta}\big(T(t)f - f\big)\right\| = ||f||_{F_\beta}.$$

Finally, A is invertible, then $R(\lambda, A)A$ is invertible and bounded. Let $\lambda = 1$ and $B = R(1, A)A$. Then $f = B^{-1}Bf$ and

$$\|f\| = \|B^{-1}Bf\| \leqslant \|B^{-1}\| \cdot \|Bf\| = M_0 \cdot \|R(1, A)Af\|$$
$$\leqslant M_0 \sup_{\lambda>0}\|\lambda^\alpha R(\lambda, A)Af\| = M_0\|f\|_{F_\alpha}.$$

Exercise 5. Prove that $|||\cdot|||_{F_\alpha}$ is a norm.

Proof. We have $|||f|||_{F_\alpha} = \sup_{\lambda>0}\|\lambda^\alpha AR(\lambda, A)f\|$ for all $f \in F_\alpha$. Checking the properties of the norm:

1) $|||f|||_{F_\alpha} = 0$ if and only if $f = 0$.

Let $f = 0$. Then $|||f|||_{F_\alpha} = \sup_{\lambda>0}\|\lambda^\alpha AR(\lambda, A)0\| = 0$.

Now let $|||f|||_{F_\alpha} = 0$. Then $\sup_{\lambda>0}\|\lambda^\alpha AR(\lambda, A)f\| = 0$. Consider the following equations $\|AR(\lambda, A)f\| = 0$ and $AR(\lambda, A)f = 0$. The semigroup is bounded, so A is invertible. Then $R(\lambda, A)f = 0$. Thus, $f = 0$.

2) $|||\beta f|||_{F_\alpha} = |\beta| \cdot |||f|||_{F_\alpha}$, for all $\beta \in \mathbb{R}$ or $\beta \in \mathbb{C}$.

Let $\beta \in \mathbb{R}$ or $\beta \in \mathbb{C}$. Then

$$|||\beta f|||_{F_\alpha} = \sup_{\lambda>0}\|\lambda^\alpha AR(\lambda, A)\beta f\| = \beta\sup_{\lambda>0}\|\lambda^\alpha AR(\lambda, A)f\| = |\beta| \cdot |||f|||_{F_\alpha}.$$

3) $|||f_1 + f_2|||_{F_\alpha} \leq |||f_1|||_{F_\alpha} + |||f_2|||_{F_\alpha}$, for all $f_1, f_2 \in F_\alpha$.

Let $f_1, f_2 \in F_\alpha$, then

$$\sup_{\lambda>0} \|\lambda^\alpha AR(\lambda, A)(f_1 + f_2)\| =$$

$$\sup_{\lambda>0} \|\lambda^\alpha AR(\lambda, A)f_1 + \lambda^\alpha AR(\lambda, A)f_2\| \leq \sup_{\lambda>0}(\|\lambda^\alpha AR(\lambda, A)f_1\| + \|\lambda^\alpha AR(\lambda, A)f_2\|)$$

$$\leq \sup_{\lambda>0} \|\lambda^\alpha AR(\lambda, A)f_1\| + \sup_{\lambda>0} \|\lambda^\alpha AR(\lambda, A)f_2\| = |||f_1|||_{F_\alpha} + |||f_2|||_{F_\alpha}.$$

Thus, $|||f_1 + f_2|||_{F_\alpha} \leq |||f_1|||_{F_\alpha} + |||f_2|||_{F_\alpha}$. Hence, $||| \cdot |||_{F_\alpha}$ is a norm.

Exercise 6. Prove that for all $\omega > \omega_0(T)$ the rescaled semigroups defined by $S_\omega(T) = \mathrm{e}^{-\omega t}T(t)$ yield the same Favard and Holder spaces with equivalent norms.

Proof. Consider the semigroups $T_1(t) = \mathrm{e}^{-\omega_1 t}T(t)$ and $T_2(t) = \mathrm{e}^{-\omega_2 t}T(t)$. It's clear that semigroups T_1 and T_2 are bounded. Define the corresponding Favard classes $F_\alpha(T_1)$ and $F_\alpha(T_2)$ with norms $\|f\|_{F_\alpha(T_1)}$ and $\|f\|_{F_\alpha(T_2)}$. Consider the norm $\|f\|_{F_\alpha(T_1)}$.

$$\|f\|_{F_\alpha(T_1)} = \sup_{t>0} \left\| \frac{1}{t^\alpha}\big(T_1(t)f - f\big) \right\| = \sup_{t>0} \left\| \frac{1}{t^\alpha}\big(\mathrm{e}^{-\omega_1 t}T(t)f - f\big) \right\|$$

$$= \sup_{t>0} \left\| \frac{1}{t^\alpha}\big(\mathrm{e}^{-\omega_1 t}\mathrm{e}^{\omega_2 t}\mathrm{e}^{-\omega_2 t}T(t)f - f\big) \right\| = \sup_{t>0} \left\| \frac{1}{t^\alpha}\big(\mathrm{e}^{(-\omega_1+\omega_2)t}T_2(t)f - f\big) \right\|$$

$$= \sup_{t>0} \left\| \frac{\mathrm{e}^{(-\omega_1+\omega_2)t}\big(T_2(t)f - f\big)}{t^\alpha} + \frac{\mathrm{e}^{(-\omega_1+\omega_2)t}f - f}{t^\alpha} \right\|$$

$$\leqslant \sup_{t>0} \left\| \frac{\mathrm{e}^{(-\omega_1+\omega_2)t}\big(T_2(t)f - f\big)}{t^\alpha} \right\| + \sup_{t>0} \left\| \frac{\mathrm{e}^{(-\omega_1+\omega_2)t}f - f}{t^\alpha} \right\|$$

The semigroups are bounded, so it's sufficient to consider $t \in (0, 1)$. Let

$\omega < \omega_1 < \omega_2$.Then

$$\sup_{t>0}\left\|\frac{\mathrm{e}^{(-\omega_1+\omega_2)t}\big(T_2(t)f-f\big)}{t^\alpha}\right\| + \sup_{t>0}\left\|\frac{\mathrm{e}^{(-\omega_1+\omega_2)t}f-f}{t^\alpha}\right\| \leqslant \mathrm{e}^{(-\omega_1+\omega_2)t}\|f\|_{F_\alpha(T_2)} + \Big(e^{(-\omega_1+\omega_2)t}-1\Big)\cdot\|f\|.$$

By Proposition 8.3. we have that $\|f\| \leqslant \mathrm{Const}\|f\|_{F_\alpha}$. Then

$$\mathrm{e}^{(-\omega_1+\omega_2)t}\|f\|_{F_\alpha(T_2)} + \Big(e^{(-\omega_1+\omega_2)t}-1\Big)\cdot\|f\| \leqslant \mathrm{e}^{(-\omega_1+\omega_2)t}\|f\|_{F_\alpha(T_2)} + \mathrm{Const}\|f\| = \mathrm{Const}\|f\|_{F_\alpha(T_2)}.$$

Vice versa the argumentation is symmetric, so norms are equivalent.