Lecture 7 — Solutions

Voronezh Team

Polyakov Dmitry, Karpikova Alina, Dikarev Yegor.

Exercise 1. Suppose $A : D(A) \to X$ is closed and $B \in \mathcal{L}(X)$ is bounded. a) Prove that the product AB with

$$D(AB) = \{f \in X : Bx \in D(A)\}.$$

is closed. b) Give an example for A and B such that BA with $D(BA) = D(A)$ is not closed.

Proof. $A : D(A) \to X$ is closed, so we have

$$x_n \in D(A), x_n \to x, Ax_n \to y \Rightarrow x \in D(A), Ax = y.$$

To prove that the product AB is closed we have to prove that

$$x_n \in D(AB), x_n \to x, ABx_n \to y \Rightarrow x \in D(AB), ABx = y.$$

The fact $x_n \in D(AB)$ implies that

$$Bx_n \in D(A). \tag{1}$$

B is linear bounded operator, therefore B is continuous. $x_n \to x$ implies

$$Bx_n \to Bx. \tag{2}$$

Thus, from (1),(2) using $ABx_n \to y$ we obtain that the product AB is closed.

Exercise 5. Suppose A is densely defined, and take $z \in \mathbb{C}$ with $Rez < 0$. Prove that $T(t) := A^{zt}, t > 0$ and $T(0) = I$ defines a strongly continuous semigroup.

Proof. Proving properties of the strongly continuous semigroup.

1) Let $t_1, t_2 > 0$. For two admissible curves γ and $\widetilde{\gamma}$ such that γ lies to the left of $\widetilde{\gamma}$ we have

$$
\begin{aligned}
T(t_1+t_2) &= A^{z(t_1+t_2)} = A^{zt_1+zt_2}\\
&= \frac{1}{2\pi \mathrm{i}}\int\limits_{\widetilde{\gamma}} \lambda^{zt_1+zt_2}R(\lambda,A)\mathrm{d}\lambda = \frac{1}{2\pi \mathrm{i}}\int\limits_{\widetilde{\gamma}} \lambda^{zt_1}\lambda^{zt_2}R(\lambda,A)\mathrm{d}\lambda - 0\\
&= \frac{1}{2\pi \mathrm{i}}\int\limits_{\widetilde{\gamma}} R(\lambda,A)\frac{1}{2\pi \mathrm{i}}\int\limits_{\gamma}\frac{\mu^{zt_1}\lambda^{zt_2}}{\mu-\lambda}\mathrm{d}\mu\mathrm{d}\lambda - \frac{1}{2\pi \mathrm{i}}\int\limits_{\gamma} R(\mu,A)\frac{1}{2\pi i}\int\limits_{\widetilde{\gamma}}\frac{\mu^{zt_1}\lambda^{zt_2}}{\mu-\lambda}\mathrm{d}\lambda\mathrm{d}\mu
\end{aligned}
$$

by the resolvent identity and by Fubini's theorem we have

$$
\begin{aligned}
&= \frac{1}{(2\pi \mathrm{i})^2}\int\limits_{\widetilde{\gamma}}\int\limits_{\gamma}\frac{\mu^{zt_1}\lambda^{zt_2}}{\mu-\lambda}(R(\lambda,A)-R(\mu,A))\mathrm{d}\mu\mathrm{d}\lambda\\
&= \frac{1}{(2\pi i)^2}\int\limits_{\widetilde{\gamma}}\int\limits_{\gamma}\mu^{zt_1}\lambda^{zt_2}R(\lambda,A)R(\mu,A)\mathrm{d}\mu\mathrm{d}\lambda = A^{zt_1}A^{zt_2} = T(t_1)T(t_2)
\end{aligned}
$$

2) Proving following properties of semigroup.

$$
T(0) = A^{z0} = \frac{1}{2\pi \mathrm{i}}\int\limits_{\gamma}\lambda^0 R(\lambda,A)\mathrm{d}\lambda = \frac{1}{2\pi \mathrm{i}}\int\limits_{\gamma} R(\lambda,A)\mathrm{d}\lambda = \frac{1}{2\pi \mathrm{i}}2\pi \mathrm{i} = I.
$$

$T(t)$ is bounded, because

$$
\|T(t)x\| = \|A^{zt}x\| \leqslant \frac{1}{2\pi \mathrm{i}}\int\limits_{\gamma}\lambda^{zt}\|R(\lambda,A)x\|\mathrm{d}\lambda \leqslant \frac{1}{2\pi \mathrm{i}}\int\limits_{\gamma}\frac{M\lambda^{zt}}{1+|\lambda|}\mathrm{d}\lambda\|x\| \leqslant C\|x\|,
$$

where C is a constant(integral converges for $z \to 0$. Finally, for the element $x \in \mathrm{D}(A)$ it suffices the strong continuity at 0 the dense subspace $\mathrm{D}(A)$. We have

$$\begin{aligned}
\|T(t)x - x\| &= \left\| \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} \lambda^{zt} R(\lambda, A)x - x \right\| \\
&= \left\| \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} \lambda^{zt} R(\lambda, A)x \mathrm{d}\lambda - \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} R(\lambda, A)x \mathrm{d}\lambda \right\| \\
&\leqslant \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} |\lambda^{zt} - 1| \|R(\lambda, A)\| \mathrm{d}\lambda \leqslant \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} \frac{M}{1+|\lambda|} |\lambda^{zt} - 1| \mathrm{d}\lambda
\end{aligned}$$

which converges to 0 as $z \to 0$. Hence, $T(t)$ is a strongly continuous semigroup.

Exercise 7. Let $\alpha \in (0,1)$. Prove that for all $\lambda > 0$ sufficiently large we have

$$\|A^{\alpha} R(-\lambda, A)\| < 1.$$

Compare this to Exercise 6.5.

Proof. By Proposition 7.9. for the sufficiently large fixed λ_0 we have

$$\begin{aligned}
\big\|A^{\alpha} R(-\lambda_0, A)\big\| &= \left\| \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} \lambda^{\alpha} R(\lambda, A) R(-\lambda_0, A) \mathrm{d}\lambda \right\| \\
&= \left\| \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} \frac{\lambda^{\alpha}}{\lambda + \lambda_0} \big(R(-\lambda_0, A) - R(\lambda, A)\big) \mathrm{d}\lambda \right\| \\
&\leqslant \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} \frac{\lambda^{\alpha}}{\lambda_0 + \lambda} \big\|R(-\lambda_0, A)\big\| \mathrm{d}\lambda + \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} \frac{\lambda^{\alpha}}{\lambda_0 + \lambda} \big\|R(\lambda, A)\big\| \mathrm{d}\lambda \\
&\leqslant \frac{\big\|R(-\lambda_0, A)\big\|}{2\pi \mathrm{i}} \int\limits_{\gamma} \frac{\lambda^{\alpha}}{\lambda_0 + \lambda} \mathrm{d}\lambda + \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} \frac{\lambda^{\alpha}}{\lambda_0 + \lambda} \big\|R(\lambda, A)\big\| \mathrm{d}\lambda \\
&\leqslant \frac{\big\|R(-\lambda_0, A)\big\|}{2\pi \mathrm{i}} \int\limits_{\gamma} \frac{\lambda^{\alpha}}{\lambda_0 + \lambda} \mathrm{d}\lambda + \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} \frac{\lambda^{\alpha}}{\lambda_0 + \lambda} \cdot \frac{M}{1+|\lambda|} \mathrm{d}\lambda.
\end{aligned}$$

Now in the first integral the integrand is an analytic function, so the first integral is equal to zero. By Assumption 7.3 we have that the resolvent is bounded, so by choosing λ_0 we can make integrand lesser than ε for all $\varepsilon > 0$. Thus, $\|A^\alpha R(-\lambda, A)\| < 1$.