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Lecture 7

Exercise 1:

a) Let $A : D(A) \to X$ be closed, $B \in \mathcal{L}(X)$. Show that AB with $D(AB) = \{x \in X | Bx \in D(A)\}$ is closed.
Let $(x_n) \subset D(AB)$ with

$$\begin{aligned} x_n &\to x, \\ ABx_n &\to y. \end{aligned}$$

Then it follows from $B \in \mathcal{L}(X)$ that

$$\begin{aligned} Bx_n &\to Bx \\ A(Bx_n) &\to y. \end{aligned}$$

Since A is closed, we obtain $Bx \in D(A)$ and $y = ABx$ and therefore AB is closed.

b) Take $X = C[0, 1]$ with operators

$$\begin{aligned} Bf &= \int_0^1 f(x)\,\mathrm{d}x \\ Af &= f', \quad D(A) = C^1[0, 1]. \end{aligned}$$

Then we obtain that

$$\begin{aligned} f_n &\to f \\ f_n =& BAf_n \to g \end{aligned}$$

and it does not follow that $f \in C^1[0, 1]$ as $(C^1[0, 1], \|.\|_\infty)$ is not closed.
Martin □

Exercise 3:
Prove that for $t \in \mathbb{R}$ and $f \in D(A^{it})$ we have $A^{it}f \in D(A^{-it})$ and $A^{-it}A^{it}f = f$.
If we want to show that $A^{it}f \in D(A^{-it})$, we need to show that $A^{-it-1}A^{it}f \in D(A)$.

$$A^{-it-1}A^{it}f \overset{\text{proposition 7.19}}{=} A^{-1}f \in D(A)$$

where we used proposition 7.19 a) with $z = -1$ and $w = it$. We obtain $A^{it} f \in D(A^{-it})$.
It follows immediately that

$$AA^{-it-1}A^{it}f = AA^{-1}f = f.$$

Martin □

Exercise 4:
We reformulate the proof of Proposition 7.21. We have

$-1 < Re\alpha - Re\beta < 0.$

$$A^{\alpha-\beta}x = \frac{\sin(\pi(\alpha-\beta))}{\pi}\int_0^\infty s^{\alpha-\beta}R(-\alpha+\beta, A)x\,\mathrm{d}s = -\frac{\sin(\pi(\alpha-\beta))}{\pi}\int_0^\infty s^{\alpha-\beta}(\alpha-\beta+A)^{-1}x\,\mathrm{d}s$$

$$\Longrightarrow \frac{\sin(\pi(\alpha-\beta))}{\pi}\int_0^\infty s^{\alpha-\beta}R(-\alpha+\beta, A)x\,\mathrm{d}s \in \overline{D(A)}^{\|\cdot\|_A} \overset{A \text{ closed}}{=} D(A) \overset{Re\beta\leq 1\& \text{ Proposition 7.20}}{\subset} D(A^\beta)$$

$$A^\alpha x \overset{\text{Proposition 7.19}}{=} A^\beta A^{\alpha-\beta}x = \frac{\sin(\pi(\alpha-\beta))}{\pi}A^\beta\int_0^\infty s^{\alpha-\beta}R(-\alpha+\beta, A)x\,\mathrm{d}s$$

$$= \frac{\sin(\pi(\alpha-\beta))}{\pi}\int_0^\infty s^{\alpha-\beta}R(-\alpha+\beta, A)A^\beta x\,\mathrm{d}s.$$

Martin □

Exercise 5:
Suppose A is densely defined, and take $z \in \mathbb{C}$ with $Rez < 0$. Prove that $T(t) := A^{zt}, t > 0$, and $T(0) = I$ is a strongly continuous semigroup.

Consider Corollary 7.15. In order to obtain the result, we show that A^{-z} is densely defined.

$D(A^{-z}) \overset{Def.7.17}{=} ran(A^z)$, and $ran(A^z)$ is X because A^z is bijective with inverse operator A^{-z} , and therefore surjective. So $D(A^{-z}) = X$, so A^{-z} is densely defined. Johannes □

Exercise 8:
In Exercise 1 of Lecture 4, we proved the equivalence of the given norms and each $X_n = D(A^n)$ is a Banach space. In Prop 4.14 we proved that $T(t)|_{X_n}$ is a semigroup of type (M, ω). It remains to show that $D(A^n)$ is invariant under the semigroup T.

Take $f \in D(A^n)$, so $A^{n-1}f \in D(A)$. Then

$$
\begin{aligned}
A^n f = A\dot{A}^{n-1} f &= lim_{h\searrow 0} \frac{T(t+h)A^{n-1}f - T(t)A^{n-1}f}{h} \\
&\overset{2.8b)}{=} lim_{h\searrow 0} \frac{A^{n-1}T(t+h)f - A^{n-1}T(t)f}{h} \\
&= lim_{h\searrow 0} \frac{T(h)A^{n-1}T(t)f - A^{n-1}T(t)f}{h}.
\end{aligned}
$$

$\Rightarrow A^{n-1}T(t)f \in D(A)$
$\Rightarrow T(t)f \in D(A^n)$, so $D(A^n)$ is T-invariant. Johannes □