Solutions to ISEM Lecture 7

Tehran Team

1 Exercise 7.1

a) Let $(f_n) \subset D(AB)$ with $f_n \longrightarrow f$ and $ABf_n \longrightarrow g$ in X for some $f, g \in X$. Then

$$(Bf_n) \subset D(A),$$

and since B is bounded, we have

$$\|Bf_n - Bf\| \leq \|B\|\|f_n - f\| \to 0 \quad \text{as} \quad n \to \infty,$$

i.e, $Bf_n \longrightarrow Bf$. On the other hand $ABf_n \longrightarrow g$, therefore $Bf \in D(A)$ and $ABf = g$, since A is closed. So AB is closed.

b) BA with $D(BA) := D(A)$ is closed if $B^{-1} \in \mathcal{L}(X)$.

2 Exercise 7.2

Let $m = (m_n)_n \subset (0, \infty)$ be a strictly increasing sequence with $\lim_n m_n = \infty$, and let M_m be the multiplication operator. Then M_m fulfills Assumption 7.3

Given $\lambda \in (-\infty, m_1)$, we form sequence $(\lambda - m)^{-1} := (\frac{1}{\lambda - m_n})_n$ and compute

$$\|(\lambda - m)^{-1}\| = \sum_{n \geq 1} \left| \frac{1}{\lambda - m_{n+1}} - \frac{1}{\lambda - m_n} \right| = \sum_{n \geq 1} \left| \int_{m_n}^{m_{n+1}} \frac{dt}{(\lambda - t)^2} \right| \leq \int_{m_1}^{\infty} \frac{dt}{|\lambda - t|^2}.$$

Since $\|M_a\| \leq K\|a\|$ for $\|a\| < \infty$, then the operator associated to $(\lambda - m)^{-1}$ is bounded and it is easily seen that this operator equals $R(\lambda, M_m) = M_{(\lambda - m)^{-1}}$. In particular $(-\infty, m_1) \subset \rho(M)$. For $\lambda = |\lambda| e^{i\theta}, 0 < |\theta| < \pi$, we obtain

$$\|\lambda R((\lambda, M)\| \leq K|\lambda|\|(\lambda - M)^{-1}\| \leq K \int_0^{\infty} \frac{|\lambda|}{|\lambda - t|^2} dt = K \int_0^{\infty} \frac{dt}{|e^{i\theta} - t|^2}.$$

This show that M_m fulfills Assumption 7.3.

4 Exercise 7.4

Let $f \in D(A^\beta)$ and $\gamma = \alpha - \beta$, then $-1 < \operatorname{Re} \gamma = \operatorname{Re} \alpha - \operatorname{Re} \beta < 0$, we obtain from (7.4) in Proposition 7.13:

$$A^\gamma f = A^{\alpha - \beta} f = -\frac{sin\pi(\alpha - \beta)}{2} \int_0^{\infty} s^{\alpha - \beta} (s + A)^{-1} f ds.$$

Since $s^{\alpha-\beta}(s+A)^{-1}f \in D(A^{\beta})$, for every $s > 0$ and since

$$\int_0^\infty s^{\alpha-\beta}(s+A)^{-1}A^{\beta} f ds,$$

is a convergent improper integral, the closedness A^{β} implies that

$$A^{\beta}A^{\alpha-\beta}f = \frac{sin\pi(\beta-\alpha)}{\pi}A^{\beta}\int_0^\infty s^{\alpha-\beta}(s+A)^{-1}f ds = \frac{sin\pi(\beta-\alpha)}{\pi}\int_0^\infty s^{\alpha-\beta}(s+A)^{-1}A^{\beta} f ds.$$

By Proposition 7.19.a) we have $A^{\alpha} = A^{\beta}A^{\alpha-\beta}$, hence the statement is proved.

5 Exercise 7.5

$$A^z = \frac{sin\pi z}{\pi}\int_0^\infty s^z R(-s,A) ds = -\frac{sinz}{\pi}\int_0^\infty s^z (s+A)^{-1} ds,$$

$$A^{zt}f - f = \frac{\sin(\pi zt)}{\pi}\int_0^\infty s^{zt} R(-s,A) ds - f$$

$$= \frac{\sin(\pi zt)}{\pi}\int_0^\infty s^{zt}\big(R(-s,A) - R(-s,I)\big) f ds$$

$$= \frac{\sin(\pi zt)}{\pi}\int_0^\infty s^{zt} R(-s,A)(I-A) f ds$$

$$\Longrightarrow \ \|A^{zt}f - f\| \le \frac{M|\sin(\pi zt)|}{\pi}\int_0^\infty \frac{|s^{zt}|}{1+s}\|R(-s,A)\|\|I - Af\| ds$$

$$\le \frac{|\sin(\pi zt)|}{\pi}\int_0^\infty \frac{|s^{zt}|}{1+s} ds\, \|I - Af\|,$$

which converges to 0 as $t \searrow 0$.

8 Exercise 7.8

Let $f \in D(A^n)$ and $T_n(t)f = T(t)f|_{X_n}$. Then $\|T_n(t)f\| = \|T(t)f\| \le Me^{\omega t}$.