Problem 1

a) Consider the sequence x_n in $D(AB)$ converging to x, such that ABx_n converges to g. As B is bounded, the sequence Bx_n is also convergent with the limit Bx. The elements x_n are from $D(AB)$, so Bx_n are from $D(A)$. Since A is closed, this yields that Bx also belongs to $D(A)$ and so x is from $D(AB)$. Moreover, ABx_n converges to ABx, i.e., $g = ABx$, which completes the proof.

Problem 2

Let $(m_n) \subseteq \mathbb{C}$ be a sequence. The operator M_m acts from l_2 onto l_2 iff $\exists K > 0$: $|m_n| \leq K$ $\forall n \in \mathbb{N}$. The inverse operator M_m^{-1} acts in the following way: $\forall (x_n) \in l_2$ $M_m^{-1}(x_n) = (\frac{x_n}{m_n})$ ant it acts from l_2 onto l_2 and is bounded iff $\exists K' > 0$: $|m_n| \geq K'$. Thus, $0 < K' \leq |m_n| \leq K$ $\forall n \in \mathbb{N}$.

Let us prove the following proposition:

Proposition

For operator M_m the following conditions are equivalent:

1. $(-\infty, 0] \subseteq \rho(M_m)$ and $||R(\lambda, M_m)|| \leq \frac{M}{1+|\lambda|}$ for all $\lambda \leq 0$ and for some $M > 0$.

2. $\exists C > 0$: $K' \geq C$ and $\forall m_n$ if $\mathrm{Re}(m_n) < 0$ then $|\mathrm{Im}(m_n)| \geq C$

Prof.

Let us suppose that the condition 2 holds. Then $\forall \lambda \leq 0$ $\forall (x_n) \in l_2$

$$||R(\lambda, M_m)(x_n)||_{l_2}^2 = \sum_{n=1}^{+\infty} \frac{|x_n|^2}{|\lambda - x_n|^2} = \sum_{n=1}^{+\infty} \frac{|x_n|^2}{(\lambda - \mathrm{Re}(m_n))^2 + (\mathrm{Im}(m_n))^2}$$

$$\leq \sum_{n=1}^{+\infty} |x_n|^2 \max\left\{\frac{1}{(\lambda - \mathrm{Re}(m_n))^2 + C^2}, \frac{1}{\lambda^2 + K'^2}\right\}.$$

Now let us consider two cases:

1. $\lambda \leq -K$

Then

$$||R(\lambda, M_m)(x_n)||_{l_2}^2 \leq ||(x_n)||_{l_2}^2 \max\left\{\frac{1}{(\lambda + K)^2 + C^2}, \frac{1}{\lambda^2 + K'^2}\right\} = \frac{||(x_n)||_{l_2}^2}{(\lambda + K)^2 + C^2}.$$

Thus, $||R(\lambda, M_m)|| \leq \frac{1}{\sqrt{(\lambda+K)^2+C^2}}$

Using the fact, that $\sqrt{(\lambda + K)^2 + C^2} \sim 1 + |\lambda|$, $\lambda \to -\infty$, there is $\tilde{M}_1 > 0$ and $\tilde{K} > K > 0$: $\forall \lambda < -\tilde{K}$ the inequality holds:

$$\frac{1}{\sqrt{(\lambda + K)^2 + C^2}} \leq \frac{\tilde{M}_1}{1 + |\lambda|}.$$

On the other hand, $\forall\lambda: \ -\tilde{K} \le \lambda \le -K$ the function $\frac{1}{\sqrt{(\lambda+K)^2+C^2}}$ is bounded. I. e., there is $\tilde{M}_2 > 0$:

$$\frac{1}{\sqrt{(\lambda+K)^2+C^2}} \le \tilde{M}_2 \le \frac{\tilde{M}_2(1+\tilde{K})}{1+|\lambda|} \quad \forall\lambda: \ -\tilde{K} \le \lambda \le -K.$$

Let us denote: $M_1 := \max\{\tilde{M}_1, \tilde{M}_2(1+\tilde{K})\}$.

2. $\lambda > -K$

In this case

$$||R(\lambda, M_m)|| \le \frac{1}{C} \le \frac{1+K}{C(1+|\lambda|)} = \frac{M_2}{1+|\lambda|},$$

where $M_2 = \frac{1+K}{C}$. Thus,

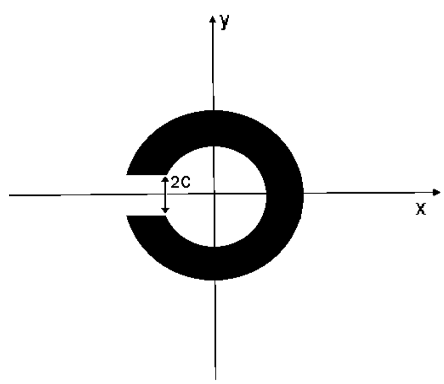
$$||R(\lambda, M_m)|| \le \frac{M}{1+|\lambda|},$$

where $M = \max\{M_1, M_2\}$.

Now let us prove that condition 1 implies condition 2. Suppose the contrary: there is a subsequence m_{n_k} such that $\mathrm{Re}(m_{n_k}) < 0$ and $\mathrm{Im}(m_{n_k}) \to 0, \ k \to +\infty$. From condition 1 we have that there is $M > 0$: $\forall\lambda \le 0$

$$||R(\lambda, M_m)|| \le M$$

Let $e_{n_k} = (x_1, x_2, \ldots)$ be the sequence from l_2 such that $x_j = 0, \ j \ne n_k$, $x_{n_k} = 1$. Let us consider the following sequence: $\lambda_k = \mathrm{Re}(m_{n_k}) < 0$. If there is $k \in \mathbb{N}$: $\mathrm{Im}(m_{n_k}) = 0$, then $R(\lambda_k, M_m)$ is not correctly defined. Otherwise, if $\mathrm{Im}(m_{n_k}) > 0 \ \forall k \in \mathbb{N}$, then $||R(\lambda_k, M_m)e_{n_k}||_{l_2} \to +\infty$. Thus, we come to the contradiction. $\square$



Picture 1.

The domain, which contains m_n is on the picture 1. The radius of the bigger circle is K and radius of the smaller is K'. Now, choosing $a > 0$ and $\theta > 0$ sufficiently small, we can construct admissible curve $\gamma = -\gamma_1 + \gamma_2$, where $\gamma_1 = se^{i\theta} + a, \ \gamma_2 = se^{-i\theta} + a, \ s \in [0, \infty)$ and determine complex powers of M_m:

$$M_m^z = \frac{1}{2\pi i} \int\limits_{\gamma} \lambda^z R(\lambda, M_m) d\lambda, \ \mathrm{Re}(z) < 0.$$

Problem 4

Observe that $-1 < \mathrm{Re}(\alpha - \beta) < 0$. Therefore from (7.4) in Proposition 7.13 we have

$$A^{\alpha-\beta} = -\frac{\sin(\pi(\alpha-\beta))}{\pi} \int\limits_0^\infty s^{\alpha-\beta}(s+A)^{-1} f ds. \tag{0.1}$$

Since $s^{\alpha-\beta}(s+A)^{-1}f \in D(A)$ for every $s > 0$ and so $s^{\alpha-\beta}(s+A)^{-1}f \in D(A^\beta)$ and since

$$\int\limits_0^\infty s^{\alpha-\beta}(s+A)^{-1}A^\beta f ds$$

is a convergent improper integral, the closedness of A^β (by Proposition 7.20 a)) implies that the right-hand side in (0.1) belongs to $D(A^\beta)$ and that

$$A^{\alpha-\beta}A^\beta f = \frac{\sin(\pi(\beta-\alpha))}{\pi} \int\limits_0^\infty s^{\alpha-\beta}(s+A)^{-1}A^\beta f ds$$
$$= \frac{\sin(\pi(\beta-\alpha))}{\pi} A^\beta \int\limits_0^\infty s^{\alpha-\beta}(s+A)^{-1} f ds$$

By Proposition 7.19. a) we have $A^\alpha = A^{\alpha-\beta}A^\beta$ for all $f \in D(A^\beta)$, which completes the proof.

Problem 5

Let us consider $A : D(A) \to X$, where $D(A)$ is a dense subset of X and operator A. satisfies Assumption 7.3. Then,

$$T(t) := A^{zt} := \int\limits_{\gamma} \lambda^z R(\lambda, A) d\lambda, \ \mathrm{Re} z < 0$$

is independent of γ by Lemma 7.8. Moreover, using Proposition 7.11, we obtain that for $z \in \mathbb{C}$: $\mathrm{Re} z < 0$, $\forall s, t > 0$

$$A^{zt}A^{zs} = A^{z(t+s)}.$$

Now let us prove that for all $f \in X$ the mapping $t \to T(t)f$ is continuous. At first, let us check that the mapping is continuous in 0. From the estimate

$$A^{zt}f - f = \frac{\sin(\pi z t)}{\pi} \int\limits_0^\infty s^{zt} R(-s, A) ds - \frac{\sin(\pi z t)}{\pi} \int\limits_0^\infty s^{zt} R(-s, I) =$$

$$= \frac{\sin(\pi z t)}{\pi} \int_0^\infty \frac{s^{zt}}{1+s} R(-s, A)(I - A) f ds.$$

we have that

$$||A^{zt} f - f|| \leq \frac{|\sin(\pi z t)|}{\pi} \int_0^\infty \frac{|s^{zt}|}{1+s} ||R(-s, A)|| ||(I - A) f|| ds \leq$$

$$\frac{|\sin(\pi z t)|}{\pi} \int_0^\infty \frac{1}{(1+s)^2} ds ||(I - A) f|| \to 0, t \searrow 0.$$

Using that for an arbitrary $t > 0$ we have:

$$A^{z(t+h)} - A^{zt} = A^{zt}(A^{zh} - 1), \;\; h > 0,$$

it is clear that the mapping $t \to A^{zt} f$ is continuous. Thus, $T(t) := A^{zt}$ is a strongly continuous semigroup.

Problem 7

Observe first that A^α is closed by Proposition 7.20 a) and for any f from the domain of the operator $A^\alpha R(-\lambda, A)$ we have that $R(-\lambda, A) f$ belongs to to $D(A)$. Fix some β such that $\alpha < \beta < 1$. Then $R(-\lambda, A) f \in D(A^\beta)$ and $D(A^\beta) \subseteq D(A^\alpha)$. Therefore by Corollary 7.26 b) there exists $K_0 \geq 0$ such that

$$\|A^\alpha R(-\lambda, A) f\| \leq K_0 \left(s^\beta \|R(-\lambda, A) f\| + s^{\beta - 1} \|A R(-\lambda, A) f\| \right)$$

holds for all $s > 0$. As

$$\|R(-\lambda, A)\| \leq \frac{M}{1 + \lambda}$$

we have

$$\begin{aligned}
\|A^\alpha R(-\lambda, A) f\| &\leq K_0 \left(s^\beta \|R(-\lambda, A) f\| + s^{\beta-1} \|(-\lambda R(-\lambda, A) - I) f\| \right) \\
&\leq K_0 \left(\frac{s^\beta M}{1+\lambda} + s^{\beta - 1} \left(\frac{M\lambda}{1+\lambda} + 1 \right) \right) \|f\| \\
&\leq K_0 \left(\frac{s^\beta M}{\lambda} + s^{\beta-1} (M+1) \right) \|f\|
\end{aligned}$$

Since this holds for all $s > 0$ we can take $s_0 > 0$ such that $s_0^{\beta - 1} < \frac{1}{2K_0(M+1)}$ and then

$$\|A^\alpha R(-\lambda, A) f\| \leq \left(K_0 \frac{s_0^\beta M}{\lambda} + \frac{1}{2} \right) \|f\|.$$

Taking $\lambda > 2 s_0^\beta K_0 M$, we obtain the assertion.

Problem 8

First statement of Proposition 7.1 was in Problems of Lecture 4. We have to prove that the Banach spaces X_n are invariant under the semigroup T. Then, for operators $T_n(t) := T(t)|_{X_n}$ all properties of semigroup are satisfied and we have only to check, that it is of type (M, ω).

We'll check that spaces X_n are invariant under $T(t)$ using induction. By Proposition 2.8 we have that $D(A)$ is invariant under semigroup $T(t)$. Assume that $D(A^n)$ is invariant under $T(t)$. By definition we have that $D(A^{n+1}) := \{f \in D(A^n) : Af \in D(A)\}$. Consider an arbitrary $f \in D(A^{n+1})$. By assumption $T(t)f \in D(A^n)$. $T(t)Af = AT(t)f \in D(A^n)$. Thus, $X_n := D(A^n)$ are invariant under $T(t)$. It remains to prove that semigroups T_n are of (M, w) type. It follows from the the density of X_n (Proposition 2.18) and definition of (M, w) type.

L'viv Team: Oleksandr Chvartatskyi, Stepan Man'ko, Nataliya Pronska.