Lecture 6 — Solutions

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Exercise 1. Let $\Omega = (0, \pi) \times (0, \pi)$ and define on $L^2(\Omega)$ the operator A as

$$Af = \Delta f, \quad D(A) := \big\{ f \in C^2(\Omega) : \text{the support of f is compact} \big\}.$$

Show that A is dissipative and its closure generates a contraction semigroup.

Proof. The support of f is a compact, so we have

$$f(0, \cdot) = f(\pi, \cdot) = f(\cdot, 0) = f(\cdot, \pi) = 0.$$

Then

$$\langle Af, f \rangle = \int_0^\pi \int_0^\pi \big(\partial_{xx} f(x, y) + \partial_{yy} f(x, y) \big) f(x, y) \mathrm{d}x \mathrm{d}y$$

$$= \int_0^\pi \left(\int_0^\pi \big(\partial_{xx} f(x, y) \big) f(x, y) \mathrm{d}x \right) \mathrm{d}y + \int_0^\pi \left(\int_0^\pi \big(\partial_{yy} f(x, y) \big) f(x, y) \mathrm{d}y \right) \mathrm{d}x$$

$$= \int_0^\pi \left(\big(\partial_x f(x, y) \big) f(x, y) \big|_0^\pi - \int_0^\pi \big(\partial_x f(x, y) \big)^2 \mathrm{d}x \right) \mathrm{d}y$$

$$+ \int_0^\pi \left(\big(\partial_y f(x, y) \big) f(x, y) \big|_0^\pi - \int_0^\pi \big(\partial_y f(x, y) \big)^2 \mathrm{d}y \right) \mathrm{d}x$$

$$= - \int_0^\pi \left(\int_0^\pi \big(\partial_x f(x, y) \big)^2 \mathrm{d}x \right) \mathrm{d}y - \int_0^\pi \left(\int_0^\pi \big(\partial_y f(x, y) \big)^2 \mathrm{d}y \right) \mathrm{d}x$$

$$= - \int_0^\pi \int_0^\pi \Big(\big(\partial_x f(x, y) \big)^2 + \big(\partial_y f(x, y) \big)^2 \Big) \mathrm{d}x \mathrm{d}y \leqslant 0.$$

By Proposition 6.5. we have that A is dissipative.

Consider $f \in C^2(\Omega)$. By Theorem 6.3. it is sufficient to prove that $ran(\lambda - A)$

is dense in X for some (hence all) $\lambda > 0$. Then closure of A generates a contraction semigroup. Consider $(\lambda - A)f$. If $f \in C^2(\Omega)$, then $(\lambda - A)f$ is linear combination of functions in $C^2(\Omega)$ and $C(\Omega)$. This space is dense in $L^2(\Omega)$. Then by theorem 6.3 we have that A generates a contraction semigroup.

Consider an arbitrary orthogonal system.

Let $f(x, y) = \alpha_{nm} \sin nx \sin my$ is orthonormal basis in $L^2(\Omega)$. Let

$$(\lambda - A)f = \big(\lambda + \alpha_{nm}(n^2 + m^2)\big) \sin nx \sin my, \quad m, n \in \mathbb{Z}.$$

This is orthogonal (orthonormal) system where $\sin nx \sin my$ is orthonormal basis. Then this system is dense in $L^2(\Omega)$. Then $ran(\lambda - A)$ is dense in $L^2(\Omega)$.

Exercise 4. Suppose that A generates a contraction semigroup and $B : D(B) \to X$ satisfies $D(A) \subseteq D(B)$, and has the following property: There is $a \in [0, \frac{1}{2})$ and $b > 0$ such that

$$\|Bx\| \leq a\|Ax\| + b\|x\| \quad \text{for all} \quad x \in D(A).$$

Prove that for large $\lambda > 0$ one has $\|BR(\lambda, A)\| < 1$.

Proof. Let $x = (\lambda I - A)^{-1}y$ or $x = R(\lambda, A)y$ and we have $D(A) \subseteq D(B)$. Consider following inequality

$$\|Bx\| = \|B(\lambda I - A)^{-1}y\| \leq a\|A(\lambda I - A)^{-1}y\| + b\|(\lambda I - A)^{-1}y\| \leq$$

$$a\|(A - \lambda I + \lambda I)(\lambda I - A)^{-1}y\| + b\|(\lambda I - A)^{-1}y\| \leq a\|y - \lambda(\lambda I - A)^{-1}y\| + \frac{b}{\lambda}\|y\|.$$

Proposition 6.2. implies the following inequality: $b\|(\lambda I - A)^{-1}y\| \leq \frac{b}{\lambda}\|y\|$. Consider the norm

$$\|\lambda(\lambda I - A)^{-1}y\| = \|\lambda \int_0^\infty T(t)e^{-\lambda t}dty\|$$

$$\leq \lambda \int_0^\infty \|T(t)\| |e^{-\lambda t}|dt\|y\| \leq \lambda \frac{\|y\|}{\lambda} = \|y\|.$$

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Inequality holds because A generates a contraction semigroup. Then let $a < \frac{1}{2}$ and for large $\lambda > 0$ from previous inequality we have

$$\|BR(\lambda, A)y\| < \frac{1}{2}\|y\| + \frac{b}{\lambda}\|y\| \leq \frac{1}{2}\|y\| + \frac{1}{2}\|y\| = \|y\|.$$

Thus, $\|BR(\lambda, A)\| < 1$.

Exercise 5. Let $X = \mathrm{C}_0(\mathbb{R})$ and $Af = f'' + f'$ with

$D(A) = \{f \in \mathrm{C}^2(\mathbb{R}) \cap X : f'' + f' \in X\}$.

Show that it generates a contraction semigroup.

Proof. Let $A = A_0 + B$, $A_0 f = f''$, $Bf = f'$ with

$D(A_0) = \{f \in \mathrm{C}^2(\mathbb{R}) \cap X : f'' \in X\}$ and

$D(B) = \{f \in \mathrm{C}^1(\mathbb{R}) \cap X : f' \in X\}$.

Note that A_0 generates Gaussian semigroup

$$\big(T_0(t)f\big)(x) := \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} f(y)\mathrm{e}^{-\frac{(x-y)^2}{4t}}\,\mathrm{d}y$$

which is a contraction (*see Prop.2.17.*) and B generates shift semigroup $\big(S(t)f\big)(s) = f(t+s)$. Then A generates semigroup T, defined as

$$\big(T(t)f\big)(x) = \big(T_0(t)S(t)f\big)(x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} f(y+t)\mathrm{e}^{-\frac{(x-y)^2}{4t}}\,\mathrm{d}y.$$

T is a contraction because T_0 is a contraction and by replacement $\tau = y + t$ we obtain that T is Gaussian semigroup. Thus, A generates a contraction semigroup.