Solutions for Lecture 6

Exercise 2.

According to Example 6.11, consider $f \in D(A)\backslash\{0\}$ and $s_0 \in [-1,0]$ such that $|f(s_0)| = \|f\|$. Then, by Example 6.8.1, we have that

$$\overline{f(s_0)} \cdot \delta_{s_0} \in J(f).$$

Consider the real-valued function

$$g(x) = Re\left(\overline{f(s_0)}f(x)\right).$$

Since $g(x)$ takes its maximum at $x = s_0$, it follows that, if $s_0 \in (-1,0)$, then

$$Re < Af, j(f) >= Re < f', \overline{f(s_0)} \cdot \delta_{s_0} >= \left(Re(\overline{f(s_0)}f)\right)'(s_0) = g'(s_0) = 0.$$

If $s_0 = -1$, then $g(x)$ takes its maximum at $s_0 = -1$. So, $g'(-1) \le 0$. If $s_0 = 0$, then

$$Re < Af, j(f) >= Re < f', \overline{f(0)} \cdot \delta_0 >= \left(Re(\overline{f(0)}f)\right)'(0) = g'(0).$$

$f \in D(A)$ implies that

$$g'(0) = Re\left(\overline{f(0)}\sum_{i=1}^{n} c_i f(-\tau_i)\right) \le \left|\overline{f(0)}\sum_{i=1}^{n} c_i f(-\tau_i)\right| \le$$

$$\sum_{i=1}^{n} |\overline{f(0)}||c_i||f(-\tau_i)| \le n\|f\| \max_{1\le i\le n} |c_i|\|f\| = n \max_{1\le i\le n} |c_i|\|f\|^2.$$

Hence, the operator A is quasi-dissipative with

$$\omega = n \max_{1\le i\le n} |c_i|.$$

Exercise 3.

First of all we set $D(M_m) = L_p(\Omega)$. For $f \in D(M_m)\backslash\{0\}$ define

$$\phi(s) = \begin{cases} \overline{f(s)} \cdot |f(s)|^{p-2} \cdot \|f\|^{2-p}, \text{ if } f(s) \neq 0 \\ 0, \text{ otherwise.} \end{cases}$$

Then, by Example 6.8.2, $\phi \in J(f) \subset L_q(\Omega) = L_p^*(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Therefore,

$$Re < Af, j(f) >= Re < Af, \phi >= Re \int_{\Omega} f(s) \cdot m(s) \cdot \phi(s) ds =$$

$$Re \int_{\Omega\backslash\{s: f(s)=0\}} f(s)m(s)\overline{f(s)}|f(s)|^{p-2}\|f\|^{2-p} ds = \|f\|^{2-p} \int_{\Omega\backslash\{s: f(s)=0\}} |f(s)|^p Re(m(s)) ds.$$

Since $f \in D(M_m)\backslash\{0\}$ can be chosen arbitrarily, operator M_m is dissipative if and only if $Re(m(s)) \le 0$ almost everywhere with respect to Ω.

Exercise 4.

Since the operator A generates a contraction semigroup $T(t), t \geq 0$, the resolvent of A can be estimated for $\lambda > 0$ as follows

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda}.$$

Proposition 2.9 and Proposition 2.26 yields, that for any $x \in D(A)$

$$AR(\lambda, A)x = R(\lambda, A)Ax \quad \text{and}$$

$$AR(\lambda, A)x = \lambda R(\lambda, A)x - x.$$

Consequently, for every $x \in X$ and $\lambda > 0$ we have that

$$R(\lambda, A)x \in D(A),$$

and, since $D(A) \subseteq D(B)$,

$$R(\lambda, A)x \in D(B).$$

Hence, for every $x \in X$

$$\|BR(\lambda, A)x\| \leq a\|AR(\lambda, A)x\| + b\|R(\lambda, A)x\| \leq$$

$$a\|\lambda R(\lambda, A)x - x\| + \frac{b}{\lambda}\|x\| \leq (2a + \frac{b}{\lambda})\|x\|.$$

Choosing $\lambda > \frac{b}{1-2a}$, we see, that $\|BR(\lambda, A)\| < 1$.

Exercise 5.

Define the following operators

$$\forall f \in D(B) = \{f \in C^1(\mathbb{R}) \cap X : \ f' \in X\} \quad Bf = f',$$

$$\forall f \in D(H) = \{f \in C^2(\mathbb{R}) \cap X : \ f' + f'' \in X\} \quad Hf = f''.$$

Then $D(H) \subset D(B)$ and $A = H + B$ with $D(H + B) = D(H) = D(A)$. In order to show, that the operator A generates a contraction, we will use Theorem 6.14.

The operator B is dissipative. Indeed, if we suppose, that $f \in D(B)\backslash\{0\}$ reaches its norm at $s_0 \in (-\infty, \infty)$, i.e., $|f(s_0)| = \|f\|$, and consider a real-valued function

$$g(x) = Re\left(\overline{f(s_0)}f(x)\right),$$

then, by Example 6.8.1,

$$Re < Bf, j(f) >= Re < f', \overline{f(s_0)} \cdot \delta_{s_0} >= \left(Re(\overline{f(s_0)}f)\right)'(s_0) = g'(s_0) = 0.$$

Function $f \in C_0(\mathbb{R})$ reaches its norm at ∞ or at $-\infty$ if and only if $f = 0$.

Consider the Gaussian semigroup T, defined on $C_0(\mathbb{R})$,

$$(T(t)f)(x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} f(y)dy, \quad \text{for } t > 0,$$

$$(T(0)f)(x) = f(x).$$

Then T is a strongly continuous contraction semigroup on $C_0(\mathbb{R})$ with generator H. ([Dun], p. 681) Since $B^2 = H$ and B generates a strongly continuous group, for $\lambda > 0$ the following holds

$$R(\lambda, H) = R(\lambda, B^2) = -R(\sqrt{\lambda}, B)R(-\sqrt{\lambda}, B).$$

([Dun], Theorem VII.9.5) Consequently, for every $f \in D(B)$

$$BR(\lambda, H)f = R(\lambda, H)Bf.$$

Choose $f \in C_0(\mathbb{R})$ arbitrarily.

Then, for any $\lambda > 0$

$$\|BR(\lambda, H)f\| = \|R(\lambda, H)Bf\| = \max_{x\in\mathbb{R}} \left| \int_0^\infty e^{-\lambda t} \left(\frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} f'(y)dy \right) dt \right| \leq$$

$$\|f'\| \int_0^\infty e^{-\lambda t} dt,$$

since

$$\frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} dy = 1.$$

Clearly,

$$\int_0^\infty e^{-\lambda t} dt = \lambda^{-1}.$$

Therefore,

$$\|BR(\lambda, H)f\| < 1$$

for

$$\lambda > \|f'\|.$$

Applying Theorem 6.14, we infer, that $A = H + B$ generates a contraction semigroup.

[Dun] N. Dunford, J.T. Schwartz - Linear Operators, General Theory (Rus), Publishers of Foreign Literature, Moscow, 1962.

Vitaliy Marchenko