

Lecture 6

Exercise 1:

We have $\Omega = (0, \pi) \times (0, \pi)$ and A is the operator on $L^2(\Omega)$ given by

$$Af = \Delta f, \quad D(A) := \{f \in C^2(\Omega) : \text{the support of } f \text{ is compact}\}.$$

By Theorem 6.3, if A is dissipative and densely defined then its closure generates a contraction semigroup.

There is a compact area $\Omega_1 \subseteq \Omega$ with $f = 0$ on $\Omega \setminus \Omega_1$, hence on Γ the boundary of Ω .

$$\begin{aligned} \langle Af, f \rangle &= \int_{\Omega_1} \Delta f(x_1, x_2) \overline{f(x_1, x_2)} d(x_1, x_2) \\ &\stackrel{\text{Green}}{=} \int_{\Gamma} \left(\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} \right) \overline{f(x_1, x_2)} n_1 d\sigma - \int_{\Omega_1} \frac{\partial f}{\partial x_1} \frac{\partial \bar{f}}{\partial x_1} + \frac{\partial f}{\partial x_2} \frac{\partial \bar{f}}{\partial x_2} d(x_1, x_2) \\ &= -\left\| \frac{\partial f}{\partial x_1} \right\|^2 - \left\| \frac{\partial f}{\partial x_2} \right\|^2 \leq 0, \end{aligned}$$

So A is dissipative.

The operator A is densely defined since $D(A)$ is dense in $L^2(\Omega)$.

By the Lumer Phillips Theorem, the closure of A generates a contraction semigroup. Johannes and Manuel \square

Exercise 2:

Let $X = C[-1, 0]$ and $0 < \tau_1 < \tau_2 < \dots < \tau_n = 1$. Consider the operator $Af := f'$ with

$$D(A) := \left\{ f \in C^1[-1, 0] : f'(0) = \sum_{i=1}^n c_i f(-\tau_i) \right\}$$

where $c_i \in \mathbb{C}$. Show that A is quasi-dissipative.

By *Proposition 6.6* and *Remark 6.7* we have to show that for every $f \in D(A)$ there exists $j(f) \in J(f)$ such that

$$\operatorname{Re} \langle Af, j(f) \rangle \leq \omega$$

for some $\omega \geq 0$.

Similarly to *Example 6.11* we take $f \in D(A)$ and $s_0 \in [-1, 0]$ such that $|f(s_0)| = \|f\|$. Then we have $\overline{f(s_0)}\delta_{s_0} \in J(f)$ and define for $x \in [-1, 0]$

$$g(x) := \operatorname{Re}(\overline{f(s_0)}f(x)).$$

Clearly, $g \in C^1[-1, 0]$,

$$\begin{aligned} g(x) &= \operatorname{Re}(\overline{f(s_0)}f(x)) \leq |\overline{f(s_0)}||f(x)| \\ &\leq |\overline{f(s_0)}|\|f\||x| \\ &\leq |\overline{f(s_0)}||f(s_0)| \\ &= |f(s_0)|^2 \\ &= g(s_0) \quad \forall x \in [-1, 0]. \end{aligned}$$

Hence g takes its maximum at $x = s_0$.

If $s_0 \in (-1, 0)$ and $j(f) = \overline{f(s_0)}\delta_{s_0}$, then

$$\operatorname{Re}\langle Af, j(f) \rangle = \operatorname{Re}\langle f', j(f) \rangle = (\operatorname{Re}\overline{f(s_0)}f)'(s_0) = g'(s_0) = 0.$$

If $s_0 = -1$ and $j(f) = \overline{f(s_0)}\delta_{s_0}$, then

$$\operatorname{Re}\langle Af, j(f) \rangle = \operatorname{Re}\langle f', j(f) \rangle = (\operatorname{Re}\overline{f(s_0)}f)'(s_0) = g'(-1) \leq 0.$$

If $s_0 = 0$ and $j(f) = \overline{f(s_0)}\delta_{s_0}$, then

$$\operatorname{Re}\langle Af, j(f) \rangle = \operatorname{Re}(\overline{f(s_0)}f'(s_0)) \stackrel{f \in D(A)}{=} \operatorname{Re}(\overline{f(s_0)} \sum_{k=1}^n c_k f(-\tau_k))$$

with $c_k \in \mathbb{C}$ and $0 < \tau_1 < \tau_2 < \dots < \tau_n = 1$. Now we define

$$h(x) = \operatorname{Im}(\overline{f(s_0)}f(x)) \tag{0.1}$$

and notice that $h \in C^1[-1, 0]$ and

$$\begin{aligned} -h(x) &= -\operatorname{Im}(\overline{f(s_0)}f(x)) \\ &\leq |\overline{f(s_0)}||f(x)| \\ &\leq |\overline{f(s_0)}|\|f\||x| \\ &\leq |\overline{f(s_0)}||f(s_0)| \\ &= g(s_0) \end{aligned}$$

for all $x \in [-1, 0]$.

Furthermore we see with $c_k = a_k + ib_k$ that

$$\operatorname{Re}(\overline{f(s_0)} \sum_{k=1}^n c_k f(-\tau_k)) = \sum_{k=1}^n \operatorname{Re}(c_k \overline{f(s_0)} f(-\tau_k))$$

$$\begin{aligned}
&= \sum_{k=1}^n \operatorname{Re}((a_k + ib_k)\overline{f(s_0)}f(-\tau_k)) \\
&= \sum_{k=1}^n \operatorname{Re}(a_k\overline{f(s_0)}f(-\tau_k) + ib_k\overline{f(s_0)}f(-\tau_k)) \\
&= \sum_{k=1}^n a_k \operatorname{Re}(\overline{f(s_0)}f(-\tau_k)) + \sum_{k=1}^n \operatorname{Re}(ib_k\overline{f(s_0)}f(-\tau_k)) \\
&= \sum_{k=1}^n a_k g(-\tau_k) + \sum_{k=1}^n b_k \operatorname{Re}(i\overline{f(s_0)}f(-\tau_k)) \\
&\leq n \cdot \max_k(|\operatorname{Re}(c_k)|) \cdot g(0) + \sum_{k=1}^n b_k \cdot -\operatorname{Im}(\overline{f(s_0)}f(-\tau_k)) \\
&\leq n \cdot \max_k(|\operatorname{Re}(c_k)|) \cdot g(0) + \sum_{k=1}^n |b_k| \cdot g(0) \\
&\leq n \cdot g(0)(\max_k(|\operatorname{Re}(c_k)|) + \max_k(|\operatorname{Im}(c_k)|)) := \omega < \infty.
\end{aligned}$$

Hence,

$$\operatorname{Re}\langle Af, j(f) \rangle \leq \omega \quad \forall x \in [-1, 0].$$

Manuel \square

Exercise 4:

Let A be the generator of a contraction semigroup, B a linear operator, $D(A) \subset D(B)$ and

$$\|Bx\| \leq a\|Ax\| + b\|x\| \quad a \in \left[0, \frac{1}{2}\right), \quad b > 0 \quad \forall x \in D(A).$$

Show that for large $\lambda > 0$ one has

$$\|BR(\lambda, A)\| < 1.$$

$$\begin{aligned}
\text{Let } x \in X : \left\| \underbrace{B R(\lambda, A)x}_{\in D(A)} \right\| &\leq a \|AR(\lambda, A)x\| + b \|R(\lambda, A)x\| \\
&\leq a \|AR(\lambda, A)x\| + \frac{b}{\lambda} \|x\| \\
&\stackrel{\text{resolvent equation}}{=} a \|AR(1, A)x + (1 - \lambda)AR(1, A)R(\lambda, A)x\| + \frac{b}{\lambda} \|x\| \\
&= a \|AR(1, A)x\| + a(1 - \lambda) \underbrace{\|AR(1, A)R(\lambda, A)x\|}_{\leq \frac{1}{\lambda} \|AR(1, A)x\|} + \frac{b}{\lambda} \|x\| \\
&\leq \underbrace{\|AR(1, A)x\|}_{\leq 2\|x\|} \left[\underbrace{a + \frac{a(1 - \lambda)}{\lambda}}_{= \frac{2a}{\lambda}} \right] + \frac{b}{\lambda} \|x\| \\
&\leq \|x\| \underbrace{\left[\frac{2a + b}{\lambda} \right]}_{< 1 \text{ for } \lambda \text{ big enough}} .
\end{aligned}$$

Martin \square