Solutions of the Exercises to Lecture 6
Team of Dresden

Exercise 1

Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded, A the operator in $L_2(\Omega)$ defined by

$$D(A) = \{f \in C^2(\Omega);\ \operatorname{spt} f \text{ compact}\},$$
$$Af = \Delta f \qquad (f \in D(A)).$$

For $f \in D(A)$ the inequality

$$\langle Af, f\rangle = \int_\Omega \Delta f\, \bar{f} = -\int_\Omega \nabla f\, \overline{\nabla f} = -\|\nabla f\|_2 \leq 0$$

holds, therefore A is dissipative (Proposition 6.5).

For $f : \Omega \to \mathbb{C}$ defined by $f(x_1, \ldots, x_n) := e^{x_1}$ one has $f \in L_2(\Omega) \cap C^2(\Omega)$, and for all $g \in D(A)$ one obtains

$$\langle Ag, f\rangle = \int_\Omega \Delta g\, f = -\int_\Omega \nabla g \nabla f = \int_\Omega g\, \Delta f = \int_\Omega g f = \langle g, f\rangle.$$

It follows that $f \perp \operatorname{ran}(I - A)$, therefore $\operatorname{ran}(I - A)^\perp \neq \{0\}$, $\overline{\operatorname{ran}(I - A)} \neq L_2(\Omega)$. The Lumer-Phillips theorem (Theorem 6.3) implies that the closure of A does not generate a contraction semigroup.

Exercise 2

Let $X = C[-1, 0]$, $n \in \mathbb{N}$, $0 < \tau_1 < \tau_2 < \ldots < \tau_n = 1$ and $c_k \in \mathbb{C}$ for $k \in \{1, \ldots, n\}$. Define the operator A in X by

$$D(A) := \{f \in C^1[-1, 0];\ f'(0) = \sum_{k=1}^n c_k f(-\tau_k)\}$$

and $Af := f'$ for all $f \in D(A)$.

We show that A is quasi-dissipative. Let $f \in D(A)$ and assume that $x \mapsto |f(x)|$ attains its maximum at $s \in [-1, 0]$. By Example 6.8, 1. the functional $\overline{f(s)}\delta_s$ is an element of $J(f)$. Consider the function $g : [-1, 0] \ni t \mapsto \operatorname{Re}(\overline{f(s)} f(t))$. Then g attains its maximum in s, $g \in C^1[-1, 0]$ and

$$g'(s) = \operatorname{Re}(\overline{f(s)} f'(s)) = \operatorname{Re}\langle Af, \overline{f(s)}\delta_s\rangle.$$

Defininig $\alpha := \sum_{k=1}^n |c_k|$, we will show that $g'(s) \leq \alpha \|f\|_\infty^2$. We have the following cases:

- $s \in (-1, 0)$. Then $g'(s) = 0$.

- $s=-1$. Then g is not increasing in -1 and thus $g'(-1) \leq 0$ follows.
- $s=0$. Since $f \in D(A)$, we can estimate

$$\begin{aligned} g'(0) &= \mathrm{Re}(\overline{f(0)} f'(0)) \\ &= \mathrm{Re}(\overline{f(0)} \sum_{k=1}^{n} c_k f(-\tau_k)) \\ &\leq \sum_{k=1}^{n} |c_k| \left| \overline{f(0)} f(-\tau_k) \right| \\ &\leq \alpha \left\| f \right\|_\infty^2 \end{aligned}$$

The proof is completed.

As an add-on, we will show that A is even a generator of a C_0-semigroup. We want to apply Theorem 6.3. Firstly, we show that A is densely defined. Consider the linear functional

$$\delta_0' : (C^1[-1,0], \|\cdot\|_\infty) \to \mathbb{C} : f \mapsto f'(0).$$

It is easy to see that δ_0' is not continuous. Consequently, the linear functional

$$\delta_0' - \sum_{k=1}^{n} c_k \delta_{-\tau_k} : (C^1[-1,0], \|\cdot\|_\infty) \to \mathbb{C}$$

is not continuous, because of the continuity of $\sum_{k=1}^{n} c_k \delta_{-\tau_k}$. Thus, the kernel of $\delta_0' - \sum_{k=1}^{n} c_k \delta_{-\tau_k}$ is dense in $(C^1[-1,0], \|\cdot\|_\infty)$ and hence in $C[-1,0]$, by the density of $C^1[-1,0]$ in $C[-1,0]$. But note that $D(A) = \ker(\delta_0' - \sum_{k=1}^{n} c_k \delta_{-\tau_k})$. Therefore A is densely defined.

Now we want to check condition (ii) in the Lumer-Phillips theorem. To this end, let $\alpha' > \alpha$. We show that $\alpha' - A$ is surjective. Let $g \in X$. We have to find $f \in D(A)$ such that

$$\alpha' f - Af = g.$$

The variation of constants formula gives the 'general solution' to the corresponding differential equation:

$$f(x) = c e^{\alpha' x} - e^{\alpha' x} \int_{-1}^{x} e^{-\alpha' \xi} g(\xi) d\xi,$$

for some $c \in \mathbb{R}$. Consider the derivative of f in 0:

$$f'(0) = \alpha' c - \alpha' \int_{-1}^{0} e^{-\alpha' \xi} g(\xi) d\xi - g(0).$$

Now, $f \in D(A)$ if and only if the following equality holds

$$\begin{aligned} f'(0) &= \sum_{k=1}^{n} c_k f(-\tau_k) \\ &= \sum_{k=1}^{n} c_k \left(c e^{-\alpha' \tau_k} - e^{-\alpha' \tau_k} \int_{-1}^{-\tau_k} e^{-\alpha' \xi} g(\xi) d\xi \right). \end{aligned}$$

This leads to the equivalent condition

$$\alpha' c - \alpha' \int_{-1}^{0} e^{-\alpha'\xi} g(\xi) d\xi - g(0) = c \sum_{k=1}^{n} c_k e^{-\alpha'\tau_k} - \sum_{k=1}^{n} c_k \left(e^{-\alpha'\tau_k} \int_{-1}^{-\tau_k} e^{-\alpha'\xi} g(\xi) d\xi \right). \quad (*)$$

Since

$$\left| \sum_{k=1}^{n} c_k e^{-\alpha'\tau_k} \right| \leq \sum_{k=1}^{n} |c_k| = \alpha < \alpha',$$

Equation $(*)$ has a unique solution $c \in \mathbb{R}$. For this choice of c, we conclude that $f \in D(A)$. Summarizing, we get that $(\alpha' - \alpha) - (A - \alpha) = \alpha' - A$ is onto. Thus, by Theorem 6.3, the dissipative operator $A - \alpha$ generates a contraction semigroup and thus by rescaling, we have that A is indeed a generator of a C_0-semigroup.

Exercise 3

We assume $(\Omega, \mathcal{A}, \mu)$ to be a $\sigma-$finite measure space and $p \in [1, \infty)$. For $m : \Omega \to \mathbb{C}$ measurable let

$$D(M_m) := \{ f \in L_p(\mu) \,|\, mf \in L_p(\mu) \}$$

and

$$M_m f := mf \quad (f \in D(M_m)).$$

Lemma 1. *If M_m is dissipative then M_m generates a contraction semigroup.*

Proof. Since M_m is densely defined and closed, it suffices to check that $R(1 - M_m)$ is dense in $L_p(\mu)$. Since by the dissipativity of M_m the operator $1 - M_m$ is injective, we conclude that

$$\mu(\{ s \in \Omega \,|\, m(s) = 1 \}) = 0.$$

For $n \in \mathbb{N}$ we define

$$\Omega_n := \{ s \in \Omega \,|\, |m(s) - 1| > n^{-1} \}.$$

Then for each $g \in L_p(\mu)$ it follows that $g\chi_{\Omega_n} \to g$ in $L_p(\mu)$ as $n \to \infty$. We set

$$f_n(s) := \begin{cases} \frac{1}{1-m(s)} g(s) & \text{if } s \in \Omega_n, \\ 0 & \text{otherwise} \end{cases}$$

and conclude that $f_n \in D(M_m)$ with $(1 - M_m) f_n = g\chi_{\Omega_n}$ for each $n \in \mathbb{N}$. This shows the density of $R(1 - M_m)$. □

Proposition 2. *M_m is dissipative if and only if* $\operatorname{Re} m \leq 0$ *a.e.*

Proof. Let us first assume that $\operatorname{Re} m \leq 0$ a.e. For $f \in D(M_m)$ we set

$$g(s) := \begin{cases} \overline{f(s)} |f(s)|^{p-2} \|f\|_{L_p(\mu)}^{2-p} & \text{if } f(s) \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

for $s \in \Omega$ and get $g \in J(f)$. Then

$$\begin{aligned}\operatorname{Re}\langle M_m f, g\rangle &= \operatorname{Re}\int_\Omega m(s)f(s)g(s)\ \mathrm{d}\mu(s)\\ &= \|f\|_{L_p(\mu)}^{2-p}\operatorname{Re}\int_\Omega m(s)|f(s)|^p\ \mathrm{d}\mu(s)\\ &= \|f\|_{L_p(\mu)}^{2-p}\int_\Omega \operatorname{Re}\ m(s)|f(s)|^p\ \mathrm{d}\mu(s)\\ &\leq 0\end{aligned}$$

and hence M_m is dissipative.
Assume now that M_m is dissipative. According to Lemma 1 and Proposition 6.6, we get

$$\forall f \in D(M_m), \phi \in J(f) : \operatorname{Re}\langle M_m f, \phi\rangle \leq 0.$$

We take sets $E_k \in \mathcal{A}$ for $k \in \mathbb{N}$ with $\mu(E_k) < \infty$ for each $k \in \mathbb{N}$ and

$$\Omega = \bigcup_{k\in\mathbb{N}} E_k.$$

Let $k \in \mathbb{N}$ and consider the set

$$\Omega_{k,n} := \{s \in E_k \,|\, |m(s)| \leq n, \operatorname{Re}\ m \geq n^{-1}\}$$

for $n \in \mathbb{N}$. Then $\chi_{\Omega_{k,n}} \in D(M_m)$ and we define

$$g_{k,n} := \chi_{\Omega_{k,n}}\mu(\Omega_{k,n})^{\frac{2-p}{p}} \in J(\chi_{\Omega_{k,n}}).$$

Thus

$$\begin{aligned}0 &\geq \operatorname{Re}\langle M_m\chi_{\Omega_{k,n}}, g_{k,n}\rangle\\ &= \mu(\Omega_{k,n})^{\frac{2-p}{p}}\operatorname{Re}\int_{\Omega_{k,n}} m(s)\ \mathrm{d}\mu(s)\\ &\geq \frac{1}{n}\mu(\Omega_{k,n})^{\frac{2}{p}}\end{aligned}$$

and hence we conclude that $\mu(\Omega_{k,n}) = 0$ for each $k, n \in \mathbb{N}$. From this we get

$$\mu(\{s \in \Omega \,|\, \operatorname{Re}\ m(s) > 0\}) \leq \sum_{k,n\in\mathbb{N}} \mu(\Omega_{k,n}) = 0,$$

which shows $\operatorname{Re}\ m \leq 0$ a.e. This completes the proof. □

Exercise 4

Let X be a Banach space, A a generator of a contraction C_0-semigroup on X, and B an operator in X satisfying $D(A) \subseteq D(B)$ and

$$\|Bx\| \leq a\|Ax\| + b\|x\| \quad (x \in D(A))$$

for some $a \in \left[0, \frac{1}{2}\right)$ and $b \in (0, \infty)$.

Then there exists $\lambda_0 > 0$ such that $\|BR(\lambda, A)\| < 1$ holds for all $\lambda > \lambda_0$.

Proof. Since A generates a contraction C_0-semigroup, it follows that

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda}$$

and therefore

$$\|AR(\lambda, A)\| = \|\lambda R(\lambda, A) - I\| \leq \lambda\|R(\lambda, A)\| + 1 \leq 2$$

for all $\lambda > 0$.

Then the assumption gives

$$\|BR(\lambda, A)x\| \leq a\|AR(\lambda, A)x\| + b\|R(\lambda, A)x\| \leq \left(2a + \frac{b}{\lambda}\right)\|x\| \quad (x \in X)$$

and so

$$\|BR(\lambda, A)\| \leq 2a + \frac{b}{\lambda}$$

for all $\lambda > 0$.

To finish the proof set $\lambda_0 := \frac{b}{1-2a} > 0$. □

Exercise 5

Let $X = C_0(\mathbb{R})$ (with the sup-norm) and $Af = f'' + f'$ with $D(A) = \left\{f \in C^2(\mathbb{R}) \cap X;\, f'' + f' \in X\right\}$. Show that A generates a contraction semigroup.

Solution: (i) Define the operators B and C in X by

$$D(B) := \left\{f \in C^2(\mathbb{R}) \cap X;\, Bf := f'' \in X\right\},$$
$$D(C) := \left\{f \in C^1(\mathbb{R}) \cap X;\, Cf := f' \in X\right\}.$$

The dissipativity of B and C is shown as in Examples 6.11 and 6.12, respectively, and the dissipativity of A is shown analogously.

(ii) In this step we show that B generates a contraction semigroup. In view of the dissipativity of B, Proposition 6.2 and Theorem 6.3 it is sufficient to show that $\operatorname{ran}(I-B) = X$.

Let $g \in X$. We have to find $f \in D(B)$ such that $f - Bf = g$. In order to find f we solve the differential equation

$$f - f'' = g.$$

The associated homogeneous equation

$$f - f'' = 0$$

has the 'general solution'

$$f(x) = c_1 e^x + c_2 e^{-x}.$$

Applying the 'variation of constants method' one obtains $c_1' = -\frac{1}{2}e^{-x}g(x)$, $c_2' = \frac{1}{2}e^x g(x)$ and the 'general solution' of the inhomogenous equation

$$f(x) = c_1 e^x + c_2 e^{-x} + \tfrac{1}{2}e^x \int_x^\infty e^{-y} g(y)\, dy + \tfrac{1}{2}e^{-x} \int_{-\infty}^x e^y g(y)\, dy.$$

In order to obtain that $f \in C_0(\mathbb{R})$ one chooses $c_1 = c_2 = 0$ and finally obtains

$$\begin{aligned} f(x) &= \tfrac{1}{2}e^x \int_x^\infty e^{-y} g(y)\, dy + \tfrac{1}{2}e^{-x} \int_{-\infty}^x e^y g(y)\, dy \\ &= \tfrac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} g(y)\, dy \\ &= \tfrac{1}{2} \int_{\mathbb{R}} e^{-|y|} g(x-y)\, dy. \end{aligned}$$

Applying the dominated convergence theorem in the last expression we obtain that $f(x) \to 0$ as $x \to \pm\infty$. This means that we have shown that $f \in D(B)$ and $(I - B)f = g$.

(We note that in fact the operator B generates the heat semigroup on $C_0(\mathbb{R})$. Showing this, however, is lengthier. Also, it is instructive to compute the resolvent of a differential operator, anyway.)

(iii) We show that $D(B) \subseteq D(C)$ and that there exist $a \in [0, 1/2)$, $b > 0$ such that

$$\|Cf\| \le a\|Bf\| + b\|f\|$$

for all $f \in D(B)$.

Let $f \in D(B)$. For $x \in \mathbb{R}$, $c > 0$ we compute

$$\begin{aligned} f(x+c) - f(x) &= \int_x^{x+c} f'(y)\, dy \\ &= f'(y)(y - (x+c))\big|_x^{x+c} - \int_x^{x+c} f''(y)(y - (x+c))\, dy \\ &= -f'(x)(-c) - \int_0^c f''(x+y)(y-c)\, dy, \end{aligned}$$

$$f'(x) = \frac{f(x+c) - f(x)}{c} + \frac{1}{c}\int_0^c f''(x+y)(y-c)\,dy.$$

The last expression shows that $f'(x) \to 0$ as $x \to \pm\infty$; hence $f \in D(C)$. It also shows that

$$\|f'\| \le \tfrac{2}{c}\|f\| + \tfrac{1}{c}\left|\int_0^c (y-c)\,dy\right|\|f''\| = \tfrac{2}{c}\|f\| + \tfrac{c}{2}\|f''\|.$$

Choosing $c = 1/2$ we obtain $\|Cf\| \le \frac{1}{4}\|Bf\| + 4\|f\|$.

(iv) Using that B generates a contraction semigroup and combining part (iii), Exercise 4 and Theorem 6.14 we obtain that $B + C$ generates a contraction semigroup. From the definition of A one obtaines that $A \supseteq B + C$, and therefore

$$I - A \supseteq I - (B + C).$$

The dissipativity of A implies that $I - A$ is injective, and $(0,\infty) \subseteq \rho(B + C)$ implies that $I-(B+C)$ is surjective; hence $I-A = I-(B+C)$, and $A = B+C$ generates a contraction semigroup.