Lecture 5 — Solutions

Voronezh Team: Polyakov Dmitry, Dikarev Yegor.

Exercise 1. Prove Proposition 5.2:

Proposition 5.2. *For an operator B with domain $D(B)$ the following statements hold.*

a) The assertions below are equivalent:

(i) Operator B is closable.

(ii) The closure of the graph of B

$$\overline{\mathrm{graph} B} := \overline{\{(f, Bf) : f \in D(B)\}} \subseteq X \times X$$

(which is a closed subspace of $X \times X$) is the graph of an operator A, i.e., $(f, g),\ (f, h) \in \overline{\mathrm{graph} B}$ implies $g = h$.

(iii) If $f_n \in D(B)$ with $f_n \to 0$ and $Bf_n \to g$, then $g = 0$.

b) If B is closable, let A be the operator from a). Then A is smallest closed extension of B.

c) Operator B is closable if and only if $\lambda - B$ is closable for $\lambda \in \mathbb{R}$. We have $\overline{\lambda - B} = \lambda - \overline{B}$.

Proof. a). Consider following proposition:

(*) $\qquad (0, y) \in \overline{\mathrm{graph} B} \Longrightarrow y = 0$

and show that each of *(i)*—*(iii)* is equivalent to *(*)*.

Remark. $\overline{\mathrm{graph} B}$ *is a graph of linear operator, i.e.* $(0, y) \in \overline{\mathrm{graph} B}$

$\Rightarrow y = 0.$

Proof. Let *(i)* is true.

If $\overline{\text{graph}B} \subset \overline{\text{graph}A}$ then $(0, y) \in \overline{\text{graph}B} \Rightarrow (0, y) \in \overline{\text{graph}A} \Rightarrow y = 0$. Thus, *(*)* is true.

And now let *(*)* is true. Let $(x, y_1) \in \overline{\text{graph}B}$ and $(x, y_2) \in \overline{\text{graph}B}$. Then (because $\overline{\text{graph}B}$ is a linear subspace) $(0, y_1 - y_2) \in \overline{\text{graph}B} \Rightarrow y_1 = y_2$. Thus, $\overline{\text{graph}B}$ is a graph of the linear operator. Thus, *(*)* is true.

Thus, *(i)* $\Leftrightarrow$ *(*)*.

(ii) $\Leftrightarrow$ *(*)* because *(i)* $\Leftrightarrow$ *(*)*: condition *(i)* is more general.

(iii) $\Rightarrow$ *(*)*: let $f_n \in D(B)$ with $f_n \to 0$ and $Bf_n \to g$. Then $(0, g) \in \overline{\text{graph}B}$ and $g = 0$.

(*) $\Rightarrow$ *(iii)*: consider $f_n \in D(B)$ with $f_n \to 0$ and $Bf_n \to g$. By *(*)* we obtain $(0, g) \in \overline{\text{graph}B}$. By remark we have that $g = 0$.

b). This statement is ***incorrect***.

Consider linear operator B with $D(B) = \{0\}$, i.e. $B0 = 0$. Clearly that B is closed. Next, consider arbitrary closed linear operator A with $D(A) \neq \{0\}$. Thus, we have that *arbitrary* A is closed extension of B.

c). Prove more general condition.

Proposition. Consider closable linear operator B and bounded linear operator A. Then $B + A$ is closable with $\text{graph}(B + A) = \big\{\big(x, (B + A)x\big) : x \in D(B + A)\big\} \subseteq X \times X$.

Proof. Let $f_n \to 0$, such that $Bf_n + Af_n \to g$, because A is bounded linear operator, thus $Af_n \to 0$, and by *(iii)* we have $Bf_n \to g \Rightarrow g = 0$.

For $A = \lambda I$, all statements hold.

Exercise 2. Prove the identity:

$$\sum_{k=0}^{\infty} \frac{n^k}{k!}(n - k)^2 = n\mathrm{e}^n$$

needed in Lemma 5.7.

Proof. We consider following identity $\mathrm{e}^n = \sum\limits_{k=0}^{\infty} \frac{n^k}{k!}$. Then

$$\sum_{k=0}^{\infty} \frac{n^k}{k!}(n-k)^2 = n^2 \sum_{k=0}^{\infty} \frac{n^k}{k!} - 2n \sum_{k=0}^{\infty} \frac{n^k}{k!}k + \sum_{k=0}^{\infty} \frac{n^k}{k!}k^2 = n^2\mathrm{e}^n - 2n \sum_{k=1}^{\infty} \frac{n^k}{(k-1)!}$$
$$+ \sum_{k=1}^{\infty} \frac{n^k}{(k-1)!}k = n^2\mathrm{e}^n - 2n^2 \sum_{k=1}^{\infty} \frac{n^{k-1}}{(k-1)!} + n + \sum_{k=2}^{\infty} \frac{n^k}{(k-1)!}(k-1+1)$$
$$= n^2\mathrm{e}^n - 2n^2\mathrm{e}^n + n + n^2 \sum_{k=2}^{\infty} \frac{n^{k-2}}{(k-2)!} + n\left(\sum_{k=1}^{\infty} \frac{n^{k-1}}{(k-1)!} - 1\right)$$
$$= -n^2\mathrm{e}^n + n + n^2\mathrm{e}^n + n\mathrm{e}^n - n = n\mathrm{e}^n.$$

Thus, we obtain our identity.

Exercise 3. Prove that in Proposition 5.5 for $\lambda, \mu > \omega$ one has $R(\lambda) = R(\lambda, B)$ and $R(\mu) = R(\mu, B)$ for the same operator B.

Proof. Consider operator $B = \lambda I - R(\lambda)^{-1}$, $B : ImR(\lambda) \subset X \to X$, and operator $C = \mu I - R(\mu)^{-1}$ (by proposition 5.5, this operators could be represented in such form). Operator B is closed because $R(\lambda)^{-1}$ is closed. By resolvent identity we obtain

$$R(\lambda) = R(\mu)\big(I + (\mu - \lambda)R(\lambda)\big),$$

then $ImR(\lambda) \subset ImR(\mu)$ and $ImR(\mu) \subset ImR(\lambda)$. Hence, $\mathrm{D}(B) = \mathrm{D}(C)$. Thus we consider next operator

$$B - C = \lambda I - R(\lambda)^{-1} - \mu I + R(\mu)^{-1}.$$

For some $y = R(\lambda)x$ using resolvent identity we have

$$(B - C)R(\lambda)x = \big(\lambda I - R(\lambda)^{-1} - \mu I + R(\mu)^{-1}\big)R(\lambda)x$$
$$= (\lambda - \mu)R(\lambda)x - x + R(\mu)^{-1}R(\mu)\big(I + (\mu - \lambda)R(\lambda)\big)x$$
$$= (\lambda - \mu)R(\lambda)x + (\mu - \lambda)R(\lambda)x = 0.$$

Thus, $B = C$. Hence, $R(\lambda) = R(\lambda, B)$ and $R(\mu) = R(\mu, B)$ for the same operator B.

Exercise 6. Do the twist in Proposition 5.8. More precisely, prove that if $F, F_n : [0, t_0] \to \mathcal{L}(X)$ are strongly continuous functions that are uniformly bounded, then the following assertions are equivalent.

(i) $F_n(t)x \to F(t)x$ uniformly on $[0, t_0]$ as $n \to \infty$ for each $x \in X$.

(ii) $F_n(t)x \to F(t)x$ uniformly on $[0, t_0]$ as $n \to \infty$ for each $x \in D$ from a dense subspace D.

(iii) $F_n(t)x \to F(t)x$ uniformly on $[0, t_0]$ as $n \to \infty$ for each compact set $K \subseteq X$.

Proof. $(ii) \Rightarrow (iii)$: D is compact $\Rightarrow \forall \varepsilon > 0$ exists finite $\frac{\varepsilon}{2M}$—net, where $M = \sup\limits_{t \geqslant 0} F(t)$, i.e. exists $n_0(\varepsilon)$ such that $\|x - x_k\| < \frac{\varepsilon}{2M}$ for all $k > n_0(\varepsilon)$. Then $\|F_n(t)x_k - F_n(t)x\| \leqslant \frac{\varepsilon}{2}$. Then consider following inequality

$$\|F_n(t)x_k - F(t)x\| = \|F_n(t)x - F_n(t)x + F_n(t)x - F(t)x\|$$
$$\leqslant \|F_n(t)x_k - F_n(t)x\| + \|F_n(t)x - F(t)x\| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Because

$\|F_n(t)x_k - F_n(t)x\| \leq \|F_n(t)\|\|x_k - x\| \leqslant M\frac{\varepsilon}{2M} = \frac{\varepsilon}{2}$ by the definition of the net and $F_n(t)$ is bounded functions;

$\|F_n(t)x - F(t)x\| \leqslant \frac{\varepsilon}{2}$, because $F_n(t)x \to F(t)x$ uniformly.

$(iii) \Rightarrow (i) : x \in K \subset X \Rightarrow x \in X$.

Then (i) holds because $F_n(t)x \to F(t)x$ uniformly on $[0, t_0]$ as $n \to \infty$ for each $x \in K$, then and for each $x \in X$.

$(i) \Rightarrow (ii)$: each $x_n \in X$ is a limit of a convergent sequence of elements from D. Thus, because D is dense in X, we have uniformly convergence in$X$.