Solutions for Lecture 5

Exercise 1. Prove Proposition 5.2

Proof: a)

(i) $\Longrightarrow$ (ii)

Since B is closable there exists a closed operator A, such that $D(B) \subseteq D(A)$ and $Ax = Bx$ for $x \in D(B)$. Consider

$$(f,g),(f,h) \in \overline{graphB} \subseteq X \times X.$$

Then there exist $f_n \in D(B), g_n, h_n$ such that

$$(f_n, g_n), (f_n, h_n) \in graphB, \quad f_n \to f,$$

$$(f_n, g_n) \to (f,g) \in \overline{graphB}, \quad (f_n, h_n) \to (f,h) \in \overline{graphB}.$$

$D(B) \subseteq D(A)$ implies that $f_n \in D(A)$. Consequently

$$g_n = Bf_n = h_n = Af_n.$$

By the closedness of A we obtain that $f \in D(A)$ and

$$Af = g = h.$$

(i) $\Longrightarrow$ (iii)

Consider $f_n \in D(B)$ with $f_n \to 0$ and $Bf_n \to g$. Closability of B means that there exists a closed operator A, such that $D(B) \subseteq D(A)$ and $Ax = Bx$ for $x \in D(B)$. Hence $f_n \in D(A)$, $f_n \to 0$, $Af_n = Bf_n \to g$. Since A is closed and assuming that A is homogeneous, we infer that $0 \in D(A)$ and

$$A(0) = 0 \cdot A(0) = g = 0.$$

(iii) $\Longrightarrow$ (i)

To proove this we need the assumption that B is additive. Define the operator A with

$$D(A) = \{f \in X : \exists f_n \in D(B) \text{ such that, if } f_n \to f \in X \text{ then}$$
$$\exists h = \lim_n Bf_n \in X\}.$$

For $f \in D(A)$ we set $Af := h$. We have that

$$D(B) \subseteq D(A) \subseteq \overline{D(B)}$$

and $Ax = Bx$ for $x \in D(B)$. It is sufficient to show that A is closed. If we consider $y_n \in D(A)$ such that $y_n \to y$ and $Ay_n \to g$ then $y \in D(A)$. Now choose $f \in D(B)$, $f_n \in D(B)$ with $f_n \to f$ and $Bf_n \to h$. Then $(f_n - f) \to 0$ and

$$A(f_n - f) = B(f_n - f) = Bf_n - Af \to h - Af.$$

Hence $h = Af$. Since $D(B) \subseteq D(A) \subseteq \overline{D(B)}$ we obtain that A is closed.

(ii) $\Longrightarrow$ (i)

Define the operator A with

$$D(A) = \{f \in X : \exists f_n \in D(B) \text{ such that, if } f_n \to f \in X \text{ then}$$
$$\exists h = \lim_n Bf_n \in X\}.$$

For $f \in D(A)$ we set $Af := h$. We have that

$$D(B) \subseteq D(A) \subseteq \overline{D(B)}$$

and $Ax = Bx$ for $x \in D(B)$. It remains to show that A is closed. Choose $f_n \in D(A)$ such that $f_n \to f$ and $Af_n \to g$. We can see, that $f \in D(A)$, $(f, Af) \in \overline{graph B}$ and $(f, g) \in \overline{graph B}$. Then

$$Af = g.$$

b)

Suppose that there exists another closed extension $\tilde{A}$ of the operator B, that is less then A, i.e.,

$$D(B) \subseteq D(\tilde{A}) \subset D(A)$$

and $\tilde{A}x = Bx$ for $x \in D(B)$. For $f \in D(\tilde{A})$ we have that $Af = \tilde{A}f$. Since A is a closed extension from a) we have that

$$D(B) \subseteq D(A) \subseteq \overline{D(B)}$$

and

$$D(A) = \{f \in X : \exists f_n \in D(B) \text{ such that, if } f_n \to f \in X \text{ then}$$
$$\exists h = \lim_n Bf_n \in X\}.$$

For $f \in D(A)$ we have $Af = h$. Hence $D(\tilde{A}) \subset D(A) \subseteq \overline{D(B)}$. It means that there exists $y \in D(A)$ such that $y \notin D(\tilde{A})$. Thats why we conclude that $\tilde{A}$ is not closed, which contradicts to the assumption. Indeed, if we consider $f_n \in D(\tilde{A})$ with $f_n \to y$, $\tilde{A}f_n = Af_n \to g$ and since A is closed, then we'll see that $y \in D(A)$, $Ay = g$, but

$$y \notin D(\tilde{A}).$$

c)

The proof of "if" part is obvious. It is sufficient to consider $\lambda = 0$.

"Only if" part: The closability of B means that there exists a closed extension A of the operator B, i.e., $D(B) \subseteq D(A)$ and $Ax = Bx$ for $x \in D(B)$. Then $\lambda - A$ is a closed extension of $\lambda - B$. Indeed,

$$D(\lambda - B) = D(B), \qquad D(\lambda - A) = D(A).$$

Hence $D(\lambda - B) \subseteq D(\lambda - A)$. Clearly, for all $x \in D(\lambda - B) = D(B)$ we have, that

$$(\lambda - A)x = \lambda x - Ax = \lambda x - Bx = (\lambda - B)x.$$

$\overline{\lambda - B} = \lambda - \overline{B}$ is obvious.

Exercise 2. Prove the identity:

$$\forall n \in \mathbb{N} \quad \sum_{k=0}^{\infty} \frac{n^k}{k!}(n-k)^2 = ne^n.$$

Proof: Since the series

$$\sum_{k=0}^{\infty} \frac{n^k}{k!}, \quad \sum_{k=0}^{\infty} \frac{kn^k}{k!}, \quad \sum_{k=0}^{\infty} \frac{k^2 n^k}{k!}$$

are convergent, we can write

$$\sum_{k=0}^{\infty} \frac{n^k}{k!}(n-k)^2 = n^2 \sum_{k=0}^{\infty} \frac{n^k}{k!} - 2n \sum_{k=0}^{\infty} \frac{kn^k}{k!} + \sum_{k=0}^{\infty} \frac{k^2 n^k}{k!} =$$

$$n^2 e^n - 2n \sum_{k=0}^{\infty} \frac{kn^k}{k!} + \sum_{k=0}^{\infty} \frac{k^2 n^k}{k!}.$$

$$\sum_{k=0}^{\infty} \frac{kn^k}{k!} = \sum_{k=1}^{\infty} \frac{n^k}{(k-1)!} = n \sum_{j=0}^{\infty} \frac{n^j}{j!} = ne^n.$$

Consequently,

$$\sum_{k=0}^{\infty} \frac{n^k}{k!}(n-k)^2 = \sum_{k=0}^{\infty} \frac{k^2 n^k}{k!} - n^2 e^n.$$

Since

$$\sum_{k=0}^{\infty} \frac{k^2 n^k}{k!} = \sum_{k=1}^{\infty} \frac{kn^k}{(k-1)!} = \sum_{k=0}^{\infty} \frac{n^{k+1}(k+1)}{k!}$$

and since all the series are convergent, we can obtain that

$$\sum_{k=0}^{\infty} \frac{k^2 n^k}{k!} - n^2 e^n = \sum_{k=0}^{\infty} \frac{n^{k+1}(k+1)}{k!} - \sum_{k=0}^{\infty} \frac{n^2 n^k}{k!} = \sum_{k=0}^{\infty} \frac{n^{k+1}(k+1-n)}{k!} =$$

$$\sum_{k=0}^{\infty} \frac{n^{k+1}k}{k!} + \sum_{k=0}^{\infty} \frac{n^{k+1}(1-n)}{k!} = \sum_{k=1}^{\infty} \frac{n^{k+1}}{(k-1)!} + \sum_{k=0}^{\infty} \frac{n^{k+1}(1-n)}{k!} =$$

$$\sum_{k=0}^{\infty} \frac{n^{k+2}}{k!} + \sum_{k=0}^{\infty} \frac{n^{k+1}(1-n)}{k!} = \sum_{k=0}^{\infty} \frac{n^{k+2} + (1-n)n^{k+1}}{k!} = n \sum_{k=0}^{\infty} \frac{n^k}{k!} = ne^n.$$

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