## Problem 1

a) Let us firstly show that $(i)$ implies $(iii)$. Denote the closure of $B$ by $\overline{B}$. Consider the sequence $(f_n)$ such that $f_n \in D(B)$ with $f_n \to 0$ and $Bf_n \to g$. Since $\overline{B}$ is the closure of $B$ we also have that $f_n \in D(\overline{B})$ and $\overline{B}f_n = Bf_n \to g$. As $\overline{B}$ is closed, $g = \overline{B}0 = 0$, which completes the proof of this part.

Next we shall prove the implication $(iii) \Rightarrow (ii)$. Suppose $(f,g),(f,h)$ belong to $\overline{\text{graph}B}$. Then there exist sequences $(f_n, g_n)$ and $(\tilde{f}_n, h_n)$ in graph$B$ such that $(f_n, g_n) \to (f,g)$ and $(\tilde{f}_n, h_n) \to (f,h)$. Note that $g_n = Bf_n$ and $h_n = B\tilde{f}_n$ and thus $f_n - \tilde{f}_n \to 0$ and $B(f_n - \tilde{f}_n) \to g - h$. By $(iii)$ this means that $g - h = 0$ or $g = h$.

It remains to show that $(ii) \Rightarrow (i)$. To do this let us consider the operator $A$ from $(ii)$ and show that it is the closure of $B$, i.e., $B$ is closable and $\overline{B} = A$. It is clear that $D(B) \subset D(A)$. If we have $f \in D(B)$ then $(f, Af)$ belongs to $\overline{\text{graph}B}$. But $(f, Bf)$ also belongs to $\overline{\text{graph}B}$. By the statement of $(ii)$ this yields $Af = Bf$ for $f \in D(B)$. Thus $A|_{D(B)} = B$. And so the proof of equivalence of the assertions is finished.

b) The operator $A$ is closed and thus it is one of the closed extensions of $B$. Consider another closed extension $S$. Then graph$B \subset$ graph$S$ and so $\overline{\text{graph}B} \subset \overline{\text{graph}S}$. But graph$A = \overline{\text{graph}B}$ and as $S$ is closed $\overline{\text{graph}S} =$ graph$S$. Therefore graph$A \subset$ graph$S$. This means that $A$ is the smallest extension of $B$.

c) Suppose $B$ is closable. Then by a) if we have $f_n \in D(B)$ with $f_n \to 0$ and $Bf_n \to g$, then $g = 0$. To show that $\lambda - B$ for $\lambda \in \mathbb{R}$ is closable, in view of a), it is enough to prove that if we have $h_n \in D(\lambda - B)$ with $h_n \to 0$ and $(\lambda - B)h_n \to h$ then $h = 0$. Clearly, $\lambda h_n \to 0$ and so $Bh_n \to h$. Observing that $D(B) = D(\lambda - B)$ we immediately obtain that $h = 0$. Thus $\lambda - B$ is closable. The reverse implication can be proved analogously.

To prove that $\overline{\lambda - B} = \lambda - \overline{B}$ observe first that

$$D(\overline{\lambda - B}) = \overline{D(\lambda - B)} = \overline{D(B)} = D(\overline{B}) = D(\lambda - \overline{B}).$$

Then consider $f \in \overline{D(B)}$. Let $h = (\overline{\lambda - B})f$ and $g = (\lambda - \overline{B})f$. We have to show that $g = h$. There exist a sequence $(f_n)$ such that $f_n \in D(B)$ and $f_n \to f$. Then $(\lambda - B)f_n \to (\overline{\lambda - B})f = h$. But also $(\lambda - B)f_n = \lambda f_n - Bf_n \to (\lambda - \overline{B})f = g$. Hence $g = h$ and so $\overline{\lambda - B} = \lambda - \overline{B}$.

## Problem 2

Let us consider a finite sum:

$$\sum_{k=0}^{N} \frac{n^k}{k!}(n-k)^2 = n^2 \sum_{k=0}^{N} \frac{n^k}{k!} - 2n^2 \sum_{k=1}^{N} \frac{n^{k-1}}{(k-1)!} + \sum_{k=1}^{N} \frac{(k-1+1)n^k}{(k-1)!} =$$

$$n^2 \sum_{k=0}^{N} \frac{n^k}{k!} - 2n^2 \sum_{k=1}^{N} \frac{n^{k-1}}{(k-1)!} + n^2 \sum_{k=2}^{N} \frac{n^{k-2}}{(k-2)!} + n \sum_{k=1}^{N} \frac{n^{k-1}}{(k-1)!}$$

Now, using that $\sum_{k=0}^{+\infty} \frac{n^k}{k!} = e^n$ and taking the limit $N \to +\infty$, we obtain that

$$\sum_{k=0}^{+\infty} \frac{n^k}{k!}(n-k)^2 = ne^n.$$

## Problem 3

Suppose, contrary to our claim, that $R(\lambda) = R(\lambda, B)$ and $R(\mu) = R(\mu, \widetilde{B})$ for different operators $B$ and $\widetilde{B}$. To be exact, assume that the operators $B := \lambda - R(\lambda)^{-1}$ and $\widetilde{B} := \mu - R(\mu)^{-1}$ are different.

Since for $\lambda, \mu > \omega$ the resolvent identity

$$R(\lambda, A_n) - R(\mu, A_n) = (\mu - \lambda) R(\lambda, A_n) R(\mu, A_n)$$

holds, by passing to the limit we obtain the equality

$$R(\lambda) - R(\mu) = (\mu - \lambda) R(\lambda) R(\mu)$$

or after rewriting

$$R(\lambda) = R(\mu)\big[1 - (\mu - \lambda)R(\mu)\big]^{-1} = \big[R(\mu)^{-1} - \mu + \lambda\big]^{-1}.$$

From this it follows that

$$B - \widetilde{B} = \lambda - R(\lambda)^{-1} - \mu + R(\mu)^{-1} = 0.$$

Above contradiction completes the proof.

## Problem 4

For each $n \in \mathbb{N}$ the operator $A_n$ is acting on elements of $l_2$ as follows

$$A_n \mathbf{x} = (m_1 x_1, ... m_n x_n, 0, ...)$$

where $\mathbf{x} = (x_1, x_2, ...)$ and a sequence $m = (m_1, m_2, ...) \subseteq \mathbb{C}$. Such operators generate semigroups $T_n$ with the same type $(M, \omega)$ if $\operatorname{Re} m_n < \omega$ for all $n \in \mathbb{N}$ and the semigroup $T_n$ generated by $A_n$ is defined as follows

$$T_n(t)\mathbf{x} = (e^{m_1 t} x_1, ..., e^{m_n t} x_n, x_{n+1}, ...).$$

Let us firstly check the assertion $(i)$.

Consider the operator $A = M_m$. Then

$$A\mathbf{x} - A_n \mathbf{x} = (0, ...0, m_{n+1} x_{n+1}, ...),$$

where the first $n$ elements are zeros. Since $\mathbf{x} \in l_2$, $A_n \mathbf{x} \to A\mathbf{x}$ if there exists $\omega_0$ such that for all $n \in \mathbb{N}$, $|m_n| < \omega_0$. In our case $D = X = l_2$. It is obvious that $(\lambda - A)D$ is $X$ for $\lambda > \omega$.

To check $(ii)$ note that

$$R(\lambda, A_n)\mathbf{x} = \big((\lambda - m_1)^{-1} x_1, ..., (\lambda - m_n)^{-1} x_n, \lambda^{-1} x_{n+1}, ...\big)$$

and

$$\|R(\lambda, A_n)\| \leq \frac{1}{\lambda - \omega}.$$

Let us show that the limit $\lim_{n\to\infty} R(\lambda, A_n) g$ for all $g \in X$ and it is equal to $R(\lambda) g = R(\lambda, A) g$. For $g \in X$ we can find $f \in X$ such that $g = (\lambda - A) f$. Set $g_n = (\lambda - A_n) f_n$. By $(i)$ we have $A_n f \to Af$ and so $g_n \to g$. Consider the following difference

$$\begin{aligned} R(\lambda, A) g - R(\lambda, A_n) g &= R(\lambda, A) g - R(\lambda, A_n)(g - g_n) - R(\lambda, A_n) g_n \\ &= f - R(\lambda, A_n)(g - g_n) - f = -R(\lambda, A_n)(g - g_n). \end{aligned}$$

Since $\|R(\lambda, A_n)\| \leq \frac{1}{\lambda - \omega}$ for all $n \in \mathbb{N}$ the last expression tends to zero as $n \to \infty$ and so

$$R(\lambda, A) g = \lim_{n\to\infty} R(\lambda, A_n) g.$$

For $\lambda > \omega$ the operator $R(\lambda) = R(\lambda, A)$ has dense range.

It remains to check $(iii)$. The semigroup $T$ generated by $A$ is defined in the following way

$$T(t)\mathbf{x} = (e^{m_1 t}x_1, e^{m_2 t}x_n, ...).$$

Let us fix an interval $[0, t_0]$. For $t \in [0, t_0]$

$$T(t)\mathbf{x} - T_n(t)\mathbf{x} = (0, ...0, (e^{m_{n+1}t} - 1)x_{n+1}, ...).$$

Since $|e^{m_k t} - 1| \leq 1 + e^{\omega t_0}$ and $\mathbf{x} \in l_2$ for every $\varepsilon > 0$ we can find $N$ such that for all $n > N$ $\|T(t)\mathbf{x} - T_n(t)\mathbf{x}\| < \varepsilon$. Thus $(iii)$ holds.

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