# Solutions to the Exercises of Lecture 5

## Exercise 1

**Exercise.** *Prove Proposition 5.2.*

(a): $(i) \Rightarrow (ii)$ : Let $(f,g),(f,h) \in \overline{\text{graph}B}$. Then there exist sequences $(f_n, Bf_n) \subseteq \text{graph}B$ and $(\varphi_n, B\varphi_n) \subseteq \text{graph}B$ such that $f_n \to f$, $Bf_n \to g$, $\varphi_n \to f$ and $B\varphi_n \to h$ as $n \to \infty$.
Let $\overline{B}$ be a closed extension of $B$, which exists by (i). Then we have $\overline{B}f_n = Bf_n \to g$ and $\overline{B}\varphi_n = B\varphi_n \to h$ as $n \to \infty$. Since $\overline{B}$ is closed, we have $f \in D(\overline{B})$ and $g = \overline{B}f = h$.
$(ii) \Rightarrow (iii)$ : Let $(f_n) \subseteq D(B)$ be a sequence with $f_n \to 0$ and $Bf_n \to g$ as $n \to \infty$. Since $(f_n, Bf_n) \subseteq \text{graph}B$, we thus have $(0,g) \in \overline{\text{graph}B}$. Let $A$ be the operator whose graph is $\overline{\text{graph}B}$, which exists by (ii). It then holds $g = A0 = 0$.
$(iii) \Rightarrow (i)$ : Define an operator $\overline{B}$ by $D(\overline{B}) = \{f \in X \mid \exists (f_n) \subseteq D(B) \colon f_n \to f$ and $Bf_n \to g \in X$ as $n \to \infty\}$ and $\overline{B}f := \lim\limits_{n\to\infty} Bf_n$.
We show that $\overline{B}$ is well-defined. Let $(f_n), (\tilde{f}_n) \subseteq D(B)$ with $f_n \to f$, $\tilde{f}_n \to f$, $Bf_n \to \overline{B}f$ and $B\tilde{f}_n \to g$ as $n \to \infty$. Then we have $B(f_n - \tilde{f}_n) = Bf_n - B\tilde{f}_n \to \overline{B}f - g$ as $n \to \infty$. Since $f_n - \tilde{f}_n \to 0$ as $n \to \infty$, (iii) implies $\overline{B}f - g = 0$, which is $\overline{B}f = g$. Hence $\overline{B}$ ist well-defined.
It is clear that $\overline{B}$ is an extension of $B$.
Finally we show that $\overline{B}$ is closed. For this let $(f_n) \subseteq D(\overline{B})$ with $f_n \to f$ and $\overline{B}f_n \to g$ as $n \to \infty$. Without loss of generality let $\|f_n - f\| \le \frac{1}{2n}$ and $\|\overline{B}f_n - g\| \le \frac{1}{2n}$ for all $n \in \mathbb{N}$. Choose $(\varphi_n) \subseteq D(B)$ such that $\|\varphi_n - f_n\| \le \frac{1}{2n}$ and $\|B\varphi_n - \overline{B}f_n\| \le \frac{1}{2n}$ for all $n \in \mathbb{N}$, which is possible due to the definition of $\overline{B}$. Then we have $\|\varphi_n - f\| \le \|\varphi_n - f_n\| + \|f_n - f\| \le \frac{1}{n}$ and $\|\overline{B}\varphi_n - g\| \le \|B\varphi_n - \overline{B}f_n\| + \|\overline{B}f_n - g\| \le \frac{1}{n}$. This means $\varphi_n \to f$ and $\overline{B}\varphi_n \to g$ as $n \to \infty$. By the definiton of $\overline{B}$ we get $f \in D(\overline{B})$. Hence $\overline{B}$ is closed.

(b): Let $\tilde{A}$ be a closed extension of $B$. For $(f,g) \in \overline{\text{graph}B}$ there is a sequence $(f_n) \subseteq D(B) \subseteq D(\tilde{A})$ such that $f_n \to f$ and $\tilde{A}f_n = Bf_n \to g$. Since $\tilde{A}$ is closed, we conclude $f \in D(\tilde{A})$ and $\tilde{A}f = g$. Therefore, $(f,g) \in \text{graph}\tilde{A}$. With the operator $A$ from (a) we thus get $\text{graph}A = \overline{\text{graph}B} \subseteq \text{graph}\tilde{A}$ and hence $A \subseteq \tilde{A}$. So $A$ is the smallest closed extension of $B$.

(c): Let $B$ be closable and $\lambda \in \mathbb{R}$. Let $(f_n) \subseteq D(\lambda - B) = D(B)$ with $f_n \to 0$ and $(\lambda - B)f_n \to g$ as $n \to \infty$. Then we have $Bf_n \to g$ as $n \to \infty$ and hence, since $B$ is closable, by the implication $(i) \Rightarrow (iii)$ of (a), $g = 0$. With the implication $(iii) \Rightarrow (i)$ of (a) we conclude that $\lambda - B$ is closable.
Analogously we deduce from the closability of $\lambda - B$ the closability of $B$.

For $f \in D(\overline{\lambda - B})$ we find a sequence $(f_n) \subseteq D(\lambda - B)$ such that $f_n \to f$ and $(\lambda - B)f_n \to \overline{\lambda - B}f$ as $n \to \infty$. Hence it follows $Bf_n \to \lambda f - \overline{\lambda - B}f$ as $n \to \infty$ and therefore $f \in D(\overline{B})$ and $\overline{B}f = \lambda f - \overline{\lambda - B}f$ since $\overline{B}$ is closed. We thus get $(\lambda - \overline{B})f = \overline{\lambda - B}f$ and conclude $\overline{\lambda - B} \subseteq \lambda - \overline{B}$. Analogously we get the other inclusion and hence have $\overline{\lambda - B} = \lambda - \overline{B}$.

**Exercise 2**

**Exercise.** *Prove the identity* $\sum_{k=0}^{\infty} \frac{n^k}{k!}(n-k)^2 = ne^n$ *needed in Lemma 5.7.*

For $n \in \mathbb{N}$ it holds

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{n^k}{k!}(n-k)^2 \\
&= n^2 + n(n-1)^2 + \sum_{k=2}^{\infty} \frac{n^k}{k!}(n^2 - 2kn + k^2) \\
&= n^2 + n(n-1)^2 + \sum_{k=2}^{\infty} \frac{n^k}{k!} \cdot n^2 - 2 \cdot \sum_{k=2}^{\infty} \frac{n^{k+1}}{(k-1)!} + \sum_{k=2}^{\infty} \frac{n^k}{(k-2)!} + \sum_{k=2}^{\infty} \frac{n^k}{(k-1)!} \\
&= n^2 + n(n-1)^2 + n^2(e^n - 1 - n) - 2n^2(e^n - 1) + n^2e^n + n(e^n - 1) \\
&= ne^n.
\end{aligned}$$

**Exercise 3**

**Exercise.** *Prove that in Proposition 5.5 for* $\lambda, \mu > \omega$ *one has* $R(\lambda) = R(\lambda, B)$ *and* $R(\mu) = R(\mu, B)$ *for the same operator* $B$.

Let $\lambda, \mu > \omega$. Recall from page 53 that $R(\lambda)$ and $R(\mu)$ are injective and have a common range $R$. Let $y \in R$ and set $x = R(\mu)^{-1}y \in X$. Beginning with the resolvent identity we then conclude

$$\begin{aligned}
&R(\lambda)x - R(\mu)x = (\mu - \lambda)R(\lambda)R(\mu)x \\
\Longrightarrow &\lambda R(\lambda)R(\mu)x - R(\mu)x = \mu R(\lambda)R(\mu)x - R(\lambda)x \\
\Longrightarrow &\lambda R(\lambda)y - y = \mu R(\lambda)y - R(\lambda)R(\mu)^{-1}y \\
\Longrightarrow &\lambda y - R(\lambda)^{-1}y = \mu y - R(\mu)^{-1}y.
\end{aligned}$$

So $B$ is independent of which $\lambda > \omega$ you use in its definition. This shows the assertion.

## Exercise 4

**Exercise.** *Consider the Banach space $X := \ell^2$ and recall that for a sequence $m \subseteq \mathbb{C}$ the multiplication operator corresponding to $m$ is denoted by $M_m$. Now for $n \in \mathbb{N}$ denote by $\mathbb{1}_{\{1,2,\dots,n\}}$ the characteristic sequence of the set $\{1, 2, \dots, n\}$. For a given sequence $m \subseteq \mathbb{C}$ define $m_n := m \cdot \mathbb{1}_{\{1,\dots,n\}}$ and $A_n := M_{m_n}$ the corresponding multiplication operators. Check the various conditions of the second Trotter-Kato theorem for this sequence of operators.*

Let a sequence $m = (y_1, y_2, \dots) \subset \mathbb{C}$ be given and $m_n = (y_1, \dots, y_n, 0, 0, \dots)$. We take a sequence $A_n = M_{m_n}$ of multiplication operators on $X = \ell^2$ with $D(A_n) = X = \ell^2$. It is clear that $A_n$ generates the strongly continuous semigroup $T_n(t)$ with $T_n(t)x = (e^{ty_1}x_1, \dots, e^{ty_n}x_n, x_{n+1}, \dots)$. We note that for the multiplication operator $A$ with $D(A) = \{x \in \ell^2 : mx \in \ell^2\}$ and $Ax = mx$, we have that $D(A)$ is dense in $X$ because it contains all the finite sequences.

To apply the second Trotter-Kato-Theorem, we need to ensure that all the semigroups are of the same type $(M, \omega)$, i.e. uniformly exponentially bounded, so for the corresponding semigroups $T_n(t)$ it holds $\|T_n(t)\| \leq Me^{\omega t}$ $(n \in \mathbb{N})$. It holds for $\|T_n\|$:

$$\|T_n(t)\| = \sup_{i\in\{1,\dots,n\}} \|T_n(t)e_i\| = \sup_{i\in\{1,\dots,n\}} |e^{ty_i}| = \sup_{i\in\{1,\dots,n\}} e^{t\Re(y_i)}$$

(with the standard basis of $\ell^2$:$\{e_i : i \in \mathbb{N}\}, e_i = (0, \dots, 0, 1, 0, \dots)$ at the i-th component)
It is now obvious that the real part of the given sequence $m$ needs to be bounded for the stability condition on the semigroups, so $\sup_k \Re(y_k) < \infty$. Then all the semigroups are uniformly bounded with exponent $C = \sup_k \Re(y_k)$, i.e. are of type $(1, C)$.

For $g \in X$ we have $f := (\lambda - m)^{-1}g \in X$ and $(\lambda - m)f = g$. So $f \in D(A)$ and $\lambda f - Af = g$ with $\lambda > C$, i.e. $(\lambda - A)D(A)$ is dense in $X$. It is clear that $A_n f \to Af$ in $X$ for $f \in D(A)$.
So the requirements of Theorem 5.11a) are fullfilled.

## Exercise 5

**Exercise.** *Let $A$ be the generator of a semigroup $T$ of type $(M, \omega)$ for some $M \geq 1$ and $\omega \in \mathbb{R}$ on a Banach space $(X, \|\cdot\|)$ and let $B$ be a bounded linear operator on $X$. Then $A + B$ with $D(A + B) = D(A)$ is a generator of a semigroup.*

Before we start the proof, we state the following lemma.

**Lemma.** *Let $(X, \|\cdot\|)$ be a Banach space and let $A$ be the generator of a semigroup $T$ of type $(M, \omega)$ for some $M \geq 1$ and $\omega \in \mathbb{R}$. Then there is an equivalent norm $|||\cdot|||$ on $X$ such that $T$ is of type $(1, \omega)$.*

*Proof.* For $x \in X$ we define $|||x||| := \sup_{s>0} \mathrm{e}^{-\omega s}\|T(s)x\|$. Clearly $|||\cdot|||$ is a norm on $X$ and we have

$$|||x||| = \sup_{s>0} \mathrm{e}^{-\omega s}\|T(s)x\| \leq \sup_{s>0} \mathrm{e}^{-\omega s} M \mathrm{e}^{\omega s}\|x\| = M\|x\|$$

for every $x \in X$. Since the function $s \mapsto \mathrm{e}^{-\omega s}\|T(s)x\|$ is continuous, we obtain

$$|||x||| \geq \|T(0)x\| = \|x\|$$

for every $x \in X$. Thus $\|\cdot\|$ and $|||\cdot|||$ are equivalent.

Finally we note that

$$\begin{aligned} |||\mathrm{e}^{-\omega t}T(t)x||| &= \sup_{s>0} \mathrm{e}^{-\omega s}\mathrm{e}^{-\omega t}\|T(s)T(t)x\| \\ &= \sup_{s>0} \mathrm{e}^{-\omega(s+t)}\|T(s+t)x\| \leq \sup_{r>0} \mathrm{e}^{-\omega r}\|T(r)x\| = |||x||| \end{aligned}$$

holds for all $x \in X$ and $t \geq 0$ so that we have

$$|||T(t)||| \leq \mathrm{e}^{\omega t} \quad \text{for all } t \geq 0,$$

as asserted. □

Now we return to the proof of Exercise 5. Our aim is to apply Chernoff's product formula as stated in Theorem 5.12.

*Proof.* As in the preceding lemma, we endow $X$ with a norm $|||\cdot|||$ such that $T$ is of type $(1, \omega)$.

Next, we define $F : [0, \infty) \to \mathcal{L}(X)$ by

$$F(t) = \mathrm{e}^{tB}T(t) \quad \text{for } t \geq 0,$$

where $\mathrm{e}^{\cdot B}$ is the semigroup of type $(1, |||B|||)$ generated by $B$, see exercise 1 of lecture 2. Note that the proof of the lemma also implies $|||B||| \leq M\|B\|$.

Then $F(0) = I$ and we have

$$|||F(t)^n||| = |||(\mathrm{e}^{tB}T(t))^n||| \leq \mathrm{e}^{(\omega+|||B|||)nt}, \tag{1}$$

for all $t \geq 0$ and $n \in \mathbb{N}$.

Moreover, for all $f \in D(A)$ it holds

$$\frac{F(h)f - f}{h} = \frac{\mathrm{e}^{hB}T(h)f - f}{h} = \mathrm{e}^{hB}\left(\frac{T(h)f - f}{h}\right) + \frac{\mathrm{e}^{hB}f - f}{h}, \quad h > 0.$$

When $h$ tends to 0 from above, the second summand converges to $Bf$ and we want to show that the first summand converges to $Af$. In fact, for $f \in D(A)$ and $h > 0$ we compute

$$\begin{aligned}
&\left|\left|\left|\mathrm{e}^{hB}\left(\frac{T(h)f - f}{h}\right) - Af\right|\right|\right| \\
&\qquad \leq \left|\left|\left|\mathrm{e}^{hB}\left(\frac{T(h)f - f}{h} - Af\right)\right|\right|\right| + |||\mathrm{e}^{hB}Af - Af||| \\
&\qquad\qquad \leq \mathrm{e}^{h|||B|||}\left|\left|\left|\frac{T(h)f - f}{h} - Af\right|\right|\right| + |||\mathrm{e}^{hB}Af - Af||| \,.
\end{aligned}$$

Since $A$ is the generator of $T$ and $\mathrm{e}^{\cdot B}$ is strongly continous, the right hand side tends to 0 as $h$ tends to 0 from above. Hence we have shown that for all $f \in D(A)$ it holds

$$\lim_{h \searrow 0} \frac{F(h)f - f}{h} = Af + Bf.$$

For $\lambda > \omega + |||B|||$ Proposition 2.26c) yields

$$|||BR(\lambda, A)||| \leq \frac{|||B|||}{\lambda - \omega} < 1.$$

By this and the closedness of $A+B$ with $D(A+B) = D(A)$ we have $\lambda \in \rho(A+B)$, where the resolvent is given by

$$R(\lambda, A + B) = R(\lambda, A) \sum_{n=0}^{\infty} (BR(\lambda, A))^n.$$

In particular $(\lambda - (A + B))D(A)$ is dense in $X$. All assumptions of Theorem 5.12 are now satisfied and thus $A + B$ generates a strongly continuous semigroup S.

It is given by

$$S(t)f = \lim_{n \to \infty} (F(t/n))^n f$$

for all $f \in X$ and $t \geq 0$. From (1) we see that $S$ is of type $(1, \omega + |||B|||)$ in $(X, |||\cdot|||)$. Moreover we conclude from

$$\|S(t)f\| \leq |||S(t)f||| \leq \mathrm{e}^{(\omega+|||B|||)t}|||f||| \leq M\mathrm{e}^{(\omega+M\|B\|)t}\|f\|$$

for all $f \in X$ that $S$ is of type $(M, \omega + M\|B\|)$ in $(X, \|\cdot\|)$. □

**Exercise 6**

**Exercise.** *Prove that if $F, F_n \colon [0, t_0] \to \mathcal{L}(X)$ are uniformly bounded functions, then the following assertions are equaivalent.*

1. *$F_n(t)x \to F(t)x$ uniformly on $[0, t_0]$ as $n \to \infty$ for each $x \in X$.*
2. *$F_n(t)x \to F(t)x$ uniformly on $[0, t_0]$ as $n \to \infty$ for each $x \in D$ from a dense subspace $D$.*
3. *$F_n(t)x \to F(t)x$ uniformly on $[0, t_0] \times K$ as $n \to \infty$ for each compact set $K \subseteq X$.*

$(i) \Rightarrow (ii)$ : This is trivial by choosing $D = X$.
$(ii) \Rightarrow (iii)$ : Let $M \geq 0$ be such that $\|F\| \leq M$ and $\|F_n\| \leq M$ for all $n \in \mathbb{N}$. Let $D \subseteq X$ be a dense subspace with $F_n(t)x \to F(t)x$ uniformly on $[0, t_0]$ as $n \to \infty$ for each $x \in D$. Let $\varepsilon > 0$. Set $B_x := B(x, \frac{\varepsilon}{3M})$ for every $x \in D$. Since $D$ is dense in $X$, we have $K \subseteq X \subseteq \bigcup_{x \in D} B_x$. The compactness of $K$ and the openness of the $B_x$ deliver that we can choose finitely many $x_1, \ldots, x_d \in D$ with $K \subseteq \bigcup_{i=1}^{d} B_{x_i}$. The uniform convergence on $D$ gives us for all $i \in \{i, \ldots, d\}$ indices $n_1, \ldots, n_d \in \mathbb{N}$ such that $\|F_n(t)x_i - F(t)x_i\| \leq \frac{\varepsilon}{3}$ for all $n \geq n_i$ and all $t \in [0, t_0]$. Set $n_0 := \max_{i=1,\ldots,d} n_i$. For every $x \in K$ let $\tilde{x}$ be an element of $\{x_1, \ldots, x_d\}$ such that $x \in B_{\tilde{x}}$. Hence we get for all $n \geq n_0$ and all $t \in [0, t_0]$

$$\begin{aligned}
&\sup_{x \in K}\|F_n(t)x - F(t)x\| \\
\leq &\sup_{x \in K}\big(\|F_n(t)x - F_n(t)\tilde{x}\| + \|F_n(t)\tilde{x} - F(t)\tilde{x}\| + \|F(t)\tilde{x} - F(t)x\|\big) \\
\leq &\sup_{x \in K}\big(M \cdot \|x - \tilde{x}\| + \|F_n(t)\tilde{x} - F(t)\tilde{x}\| + M \cdot \|\tilde{x} - x\|\big) \\
\leq &\sup_{x \in K}\left(M \cdot \frac{\varepsilon}{3M} + \frac{\varepsilon}{3} + M \cdot \frac{\varepsilon}{3M}\right) = \varepsilon.
\end{aligned}$$

This shows that $F_n(t)x$ converges to $F(t)x$ uniformly on $[0, t_0] \times K$ as $n \to \infty$.
$(iii) \Rightarrow (i)$ : For each $x \in X$ choose $K_x = \{x\}$. Then $K_x$ is compact and the assertion follows.

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