Lecture 4. Exercise 1

Solution from team Wuppertal

We first give a solution of Exercise 1 where we do not require A to be the generator of a semigroup, but instead only suppose that it has a non-empty resolvent set. Then we give a counterexample showing that this assumption cannot be further relaxed.

Lemma. Let A be a closed operator on the Banach space X with $\rho(A) \neq \varnothing$. Then there are constants $c_1, \dots, c_n$ such that

$$\|A^k x\| \leq c_k(\|x\| + \|A^n x\|), \quad \text{for all} \quad x \in D(A^n),\ k = 1, \dots, n.$$

Proof. Let $\lambda \in \rho(A)$. From

$$(A-\lambda)^{-1}A = I + \lambda(A-\lambda)^{-1} = A(A-\lambda)^{-1}$$

we have that for $0 \leq k \leq m$,

$$(A-\lambda)^{-m}A^k = (A-\lambda)^{-m+k}\big((A-\lambda)^{-1}A\big)^k$$

is a bounded operator (on $D(A^k)$ with respect to the norm $\|\cdot\|$). For $1 \leq k < n$ and $x \in D(A^n)$ we have

$$\begin{aligned}
A^k x &= (A-\lambda)^{-n+k}(A-\lambda)^{n-k}A^k x \\
&= (A-\lambda)^{-n+k}\sum_{j=0}^{n-k}\binom{n-k}{j}A^{n-k-j}(-\lambda)^j A^k x \\
&= (A-\lambda)^{-n+k}A^n x + \sum_{j=1}^{n-k}\binom{n-k}{j}(-\lambda)^j(A-\lambda)^{-n+k}A^{n-j}x
\end{aligned}$$

and hence

$$\begin{aligned}
\|A^k x\| \leq{}& \|(A-\lambda)^{-n+k}\|\|A^n x\| \\
&+ \sum_{j=1}^{n-k}\binom{n-k}{j}|\lambda|^j\|(A-\lambda)^{-n+k}A^{n-k-j+1}\|\|A^{k-1}x\|.
\end{aligned}$$

So there are constants c_{1k}, c_{2k} such that

$$\|A^k x\| \leq c_{1k}\|A^{k-1}x\| + c_{2k}\|A^n x\|, \quad x \in D(A^n),\ 1 \leq k < n.$$

Using these estimates repeatedly, we obtain the claim. $\square$

Exercise 1. Let A be closed with $\rho(A) \neq \varnothing$. Then the two norms

$$\|x\|_n = \|x\| + \|A^n x\|,$$
$$|||x|||_n = \|x\| + \|Ax\| + \cdots + \|A^n x\|$$

on $D(A^n)$ are equivalent and turn $D(A^n)$ into a Banach space.

Proof. Obviously $\|x\|_n \leq |||x|||_n$. By the lemma we also have a constant c such that $|||x|||_n \leq c\|x\|_n$. So the two norms are equivalent.

To see now that $D(A^n)$ together with its graph norm $\|\cdot\|_n$ is a Banach space, we show that A^n is a closed operator: Let $x_m \in D(A^n)$, $x_m \to x$, $A^n x_m \to y$ in X. The lemma implies that each $(A^k x_m)_m$ is a Cauchy sequence in X, hence it converges. Since A is closed, we obtain inductively $x \in D(A^k)$, $A^k x_m \to A^k x$ for all $k \leq n$; in particular A^n is closed. □

The following example features a closed operator A with $\rho(A) = \varnothing$, for which the norms $\|\cdot\|_2$ and $|||\cdot|||_2$ are not equivalent.

Example. On $X = \ell^2$ we want to consider an operator A which is block diagonal with 2×2 blocks. To this end we write

$$X = \ell^2(\mathbb{C}^2) = \Big\{ (u_n)_{n\in\mathbb{N}} \,\Big|\, u_n \in \mathbb{C}^2, \sum_n \|u_n\|^2 < \infty \Big\}.$$

Let then A be given by

$$A(u_n)_n = (A_n u_n)_n \qquad A_n = \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix},$$
$$D(A) = \big\{ (u_n) \in \ell^2(\mathbb{C}^2) \,\big|\, (A_n u_n) \in \ell^2(\mathbb{C}^2) \big\}.$$

Then it is straight forward to show that A is closed and that $\rho(A) = \varnothing$. The latter follows from the fact that for $\lambda \neq 0$

$$(A_n - \lambda)^{-1} = \begin{pmatrix} -\frac{1}{\lambda} & -\frac{n}{\lambda^2} \\ 0 & -\frac{1}{\lambda} \end{pmatrix};$$

hence $\sup_n \|(A_n - \lambda)^{-1}\| = \infty$ and so $(A - \lambda)^{-1}$ is not bounded.

Now consider the sequence (x_k) in X given by

$$x_k = (u_n^{(k)}) \quad \text{where} \quad u_k^{(k)} = \begin{pmatrix} 0 \\ 1/k \end{pmatrix} \quad \text{and} \quad u_n^{(k)} = 0 \quad \text{for} \quad n \neq k.$$

Then $x_k \in D(A^2)$, $\|x_k\| = 1/k$, $\|Ax_k\| = 1$ and $A^2 x_k = 0$. Hence $x_k \to 0$ with respect to $\|\cdot\|_2$, but not with respect to $|||\cdot|||_2$. Consequently, $\|\cdot\|_2$ and $|||\cdot|||_2$ are not equivalent.