Lecture 4 — Solutions

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Exercise 1. Let A be the generator of a semigroup, and consider the space $X_n = \mathrm{D}(A^n)$ with the graph norm.

a) For $n \in \mathbb{N}$ and $x \in \mathrm{D}(A^n)$ define $|||x||| := \|x\| + \|Ax\| + ... + \|A^n x\|$. Prove that $|||\cdot|||$ and $\|\cdot\|_n$ are equivalent norms.

Proof. Suppose A is invertible operator. Consider operator $A_0 = A - \lambda_0 I, \lambda_0 \in \rho(A)$. A_0 is a generator of a semigroup $T(t)\mathrm{e}^{-\lambda_0 t}$. First we prove the equivalence of the norms $\|\cdot\|_n$ and $|||\cdot|||$ for the operator A_0, i.e.

$$m(\|x\| + \|A_0^n x\|) \leq \|x\| + \|A_0 x\| + \cdots + \|A_0^n x\| \leq M(\|x\| + \|A_0^n x\|).$$

1) $\|x\| + \|A_0^n x\| \leq \|x\| + \|A_0 x\| + \cdots + \|A_0^n x\|$, and $m = 1$.

2)

$$\|x\| + \|A_0 x\| + \cdots + \|A_0^n x\| \leq \|x\| + \|A_0^n (A_0^{-n+1} x)\| + \|A_0^n (A_0^{-n+2} x)\| + \cdots + \|A_0^n x\| \leq$$
$$\leq \|x\| + \|A_0^{-n+1}\| \|A_0^n x\| + \|A_0^{-n+2}\| \|A_0^n x\| + \cdots + \|A_0^n x\| \leq$$
$$\leq \|x\| + \|A_0^n x\| \left(\sum_{k=1}^{n} \|A_0^{-k+1}\| \right) \leq M(\|x\| + \|A_0^n x\|),$$

where $M = \max\{1, \sum\limits_{k=1}^{n} \|A_0^{-k+1}\|\}$. Thus, norms are equivalent.

Prove the equivalence of the norms $\|x\| + \|A^n x\|$ and $\|x\| + \|A_0^n x\|$, and also equivalence of the norms $\|x\| + \|A_0 x\| + \cdots + \|A_0^n x\|$ and $\|x\| + \|Ax\| + \cdots + \|A^n x\|$. The proof of these facts and proof the equivalence of the norms $\|x\| + \|A_0^n x\|$ and $\|x\| + \|A_0 x\| + \cdots + \|A_0^n x\|$ of the initial statement are equivalent.

$$\|x\| + \|A^n x\| = \|x\| + \|A^n (A - \lambda_0 I)^{-n} (A - \lambda_0 I)^n x\| =$$
$$= \|x\| + \|(A - \lambda_0 I + \lambda_0 I)^n (A - \lambda_0 I)^{-n} (A - \lambda_0 I)^n x\| =$$
$$= \|x\| + \left\| \sum_{k=0}^{n} C_n^k \lambda_0^k (A - \lambda_0 I)^{n-k} (A - \lambda_0 I)^{-n} (A - \lambda_0 I)^n x \right\| \leq$$
$$\leq M_1 (\|x\| + \|(A - \lambda_0 I)^n x\|) = M_1 (\|x\| + \|A_0^n x\|),$$

where $M_1 = \max\{1, \| \sum_{k=0}^{n} C_n^k \lambda_0^k (A - \lambda_0 I)^{-k} \|\}$.

And the second estimate is $m(\|x\| + \|A_0^n x\|) \leq \|x\| + \|A^n x\|$ or $\|A_0^n x\| \leq \|A^n x\|$. Consider following equalities

$$\|A^n x\| = \|(A_0 + \lambda_0 I)^n x\| = \|A_0^n x + \sum_{k=1}^{n} C_n^k A_0^{n-k} \lambda_0^k x\| \geq \|A_0^n x\|,$$

thus, norms $\|x\| + \|A^n x\|$ and $\|x\| + \|A_0^n x\|$ are equivalent, similar to the equivalence of norms $\|x\| + \|A_0 x\| + \cdots + \|A_0^n x\|$ and $\|x\| + \|Ax\| + \cdots + \|A^n x\|$. Therefore, norms $||| \cdot |||$ and $\| \cdot \|_n$ are equivalent.

b) Prove that X_n is a Banach space.

Proof. Let $x_n \to x$ as $n \to \infty$, as norm $||| \cdot |||$. A is a generator of semigroup, thus X_n would be a Banach space if $x \in \mathrm{D}(A^n)$.

A is a generator of semigroup, therefore A is closed operator. Hence $Ax_k \to y_1, x \in \mathrm{D}(A)$. After that $A^2 x_k = A(Ax_k) \to Ay_1$ and $Ay_1 \to y_2$, and $x \in \mathrm{D}(A^2)$. Similarly $A^n x_k = A(A^{n-1} x_k) \to y_n$, and $x \in \mathrm{D}(A^n)$. Hence X_n is a Banach space.

Exercise 2. Let $X = l^2$ and $m = (m_n)$ be a sequence with $Re m_n \leq 0$. Consider the semigroup T generated by the multiplication operator $A = M_m$ and define the Crank–Nicolson method as

$$F(h) = \left(I + \frac{h}{2} A\right) \left(I - \frac{h}{2} A\right)^{-1}.$$

a) Show that it is stable.

Proof. Prove $\|F(h)^n\| \leq M$.

We note that a operator in l^2 on sequence α_k is bounded if and only if α_k is bounded sequence. Define the operator $A = M_n$ as follows $Ax = y = (m_n x_n) \in l^2$. Note that $(Ax, x) \leq 0$. Indeed, $(Ax, x) = \sum\limits_{n=1}^{\infty} m_n x_n \overline{x_n} = \sum\limits_{n=1}^{\infty} m_n |x_n|^2 \leq 0, x \in \mathrm{D}(A)$. $F(h)$ determined as $((1+\frac{h}{2}m_n)^n(1-\frac{h}{2}m_n)^{-n}x_k)$.
Consider following statement

$$\left|\frac{(1+\frac{h}{2}m_k)^n}{(1+\frac{h}{2}m_k)^n}\right| x_k = \frac{|1+\frac{h}{2}(Rem_k + iImm_k)|^n}{|1-\frac{h}{2}(Rem_k + iImm_k)|^n} x_k.$$

Let $h \leq \frac{1}{2}$, and using the condition $Rem_k \leq 0$ we obtain following estimate

$$\|F(h)^n x_k\| = \sup_{k\geq 1}\left|\frac{(1+\frac{h}{2}(Rem_k + iImm_k))^n}{(1-\frac{h}{2}(Rem_k + iImm_k))^n}\right| \|x_k\| \leq$$

$$\leq \sup_{k\geq 1}\left|\frac{(1+\frac{h}{2}Rem_k)^2 + \frac{(Imm_k h)^2}{4}}{(1-\frac{h}{2}Rem_k)^2 + \frac{(Imm_k h)^2}{4}}\right|^{\frac{n}{2}} \|x_k\| \leq \left|\frac{(1+\frac{1}{4}Rem_k)^2 + \frac{(Imm_k h)^2}{4}}{(1-\frac{1}{4}Rem_k)^2 + \frac{(Imm_k h)^2}{4}}\right|^{\frac{n}{2}} \|x_k\| \leq$$

$$\leq \left|\frac{1 + \frac{(Imm_k h)^2}{4}}{1 + \frac{(Imm_k h)^2}{4}}\right|^{\frac{n}{2}} \|x_k\| = \|x_k\|.$$

Thus, $\|F(h)^n\| \leq M$, where $M = 1$.

b) Show that it is consistent.

Proof. This sequence (m_k) forms strongly continuous semigroup $T(t)x = (\mathrm{e}^{m_n t} x_k)$ for which $\|T(t)\| = \sup_{k\geq 1} |\mathrm{e}^{m_k t}| = \sup_{k\geq 1} |\mathrm{e}^{Rem_k t}| \leq 1$. Given $f = (f_k)$ for which $\sum\limits_{k\geq 1} |m_k|^2 |f_k|^2 < \infty$. And, finally, consider a limit

$$\lim_{h\to 0} \frac{F(h)T(t)f - T(t+h)f}{h} = \lim_{h\to 0} \frac{\frac{1+\frac{h}{2}m_k}{1-\frac{h}{2}m_k}\mathrm{e}^{m_k t} f_k - \mathrm{e}^{m_k(t+h)} f_k}{h}$$

Prove the estimation of the next norm.

$$\sum_{k\geq 1}\left|\frac{\frac{1+\frac{h}{2}m_k}{1-\frac{h}{2}m_k}\mathrm{e}^{m_k t}f_k-\mathrm{e}^{m_k(t+h)}f_k}{h}\right|^2\leq\sum_{k\geq 1}|f_k|^2|\mathrm{e}^{m_k t}|^2\left|\frac{\frac{1+\frac{h}{2}m_k}{1-\frac{h}{2}m_k}-\mathrm{e}^{m_k h}}{h}\right|^2\leq$$

$$\leq\sum_{k\geq 1}|f_k|^2|\mathrm{e}^{m_k t}|^2|m_k|^2\left|\frac{\frac{1+\frac{h}{2}m_k}{1-\frac{h}{2}m_k}-\mathrm{e}^{m_k h}}{m_k h}\right|^2\to 0,$$

as all factors are bounded and

$$\frac{\frac{1+h}{1-h}-\mathrm{e}^h}{h}=\frac{1-\mathrm{e}^h}{h}+\frac{2h}{h(1-h)}\to 0.$$

Thus,

$$\lim_{h\to 0}\frac{F(h)T(t)f-T(t+h)f}{h}=\lim_{h\to 0}\frac{\frac{1+\frac{h}{2}m_k}{1-\frac{h}{2}m_k}\mathrm{e}^{m_k t}f_k-\mathrm{e}^{m_k(t+h)}f_k}{h}=0.$$

Exercise 4. Solve the exercises in the appendix.

Appendix B.

Exercise 1. Prove first- and second-order consistency of the explicit Euler method and the explicit method of Runge, respectively.

Proof. Consider

$$\psi_n^{(1)}=-\frac{y(t_{n+1})-y(t_n)}{h_n}+f(t_n,y(t_n)).$$

Estimate the function $\psi_n^{(1)}$ when $h_n\to 0$. Using Taylor series, we obtain $y_{n+1}=y(t_n+h_n)=y(t_n)+h_n y'(t_n)+\frac{h_n^2}{2}y''(t_n)+\ldots$ $\frac{y_{n+1}-y_n}{h_n}=y'(t_n)+\mathcal{O}(H_n)$. For n^{th} we can rewrite our problem as $\frac{y_{n+1}-y_n}{h_n}=y'(t_n)$. Using $y'(t_n)=f(t_n,y(t_n))$ we obtain that

$$-\frac{y_{n+1}-y_n}{h_n}+f(t_n,y(t_n))=\mathcal{O}(h_n).$$

In the last equation LHS $= \psi_n^{(1)}$, thus explicit Euler method has first order of consistency.

Exercise 2. Construct Euler's number e with the help of Euler's method. *Hint: Consider the differential equation* $y' = y$ *with the initial condition* $y(0) = 1$.

Proof. The Euler's method usually written as

$$y_{n+1} = y_n + h_n f(t_n, y_n).$$

Consider the differential equation $y' = y$, where $f(t_n, y_n) = y$ and $y_0 = y(0) = 1$. We obtain the following approximations

$$y_1 = y_0 + hf(t_0, y_0) = 1 + hy_0 = 1 + h, \quad y_2 = y_1 + hy_1 = 1 + h + h + h^2 = (1+h)^2,$$
$$y_3 = y_2 + hy_2 = (1+h)^2 + h(1+h)^2 = (1+h)^3.$$

Continuing in this way we obtain $y_n = (1+h)^n$. For $h = \frac{1}{n}$ and for $n \to \infty$, obtain $\lim_{n\to\infty} y_n = \lim_{n\to\infty}(1 + \frac{1}{n})^n = \mathrm{e}$.

Exercise 5. Derive the Butcher Tableau for the Crank-Nicolson scheme.

Proof. The implicit method is

$$y_{n+1} = y_n + \frac{h_n}{2}\big(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\big).$$

The Butcher Tableau for this scheme is

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array}$$

In our scheme we have $y_{n+1} = y_n + h(b_1 k_1 + b_2 k_2)$, so $b_1 = b_2 = 1/2$. Next, $k_1 = f(t_n + 0, y_n + 0)$, so $c_1 = 0$ and $a_{11} = a_{12} = 0$. Finally, $k_2 = f\left(t_n + h_n, y_n + h_n\left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)\right)$, thus, $c_1 = 1$ and $a_{21} = a_{22} = 1/2$.