Solutions to ISEM Lecture 4

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Exercise 4.1

a. $|||x||| = \|x\| + \|Ax\| + \cdots + \|A^n x\| = \|x\| + \|A^n x\| + \|Ax\| \ldots + \|A^{n-1}x\|$

$$= \|x\|_n + \|Ax\| \ldots + \|A^{n-1}x\| = \|x\|_n + M \quad \text{where } M > 0,$$
$$\Longrightarrow \quad \|x\|_n < |||x||| \,.$$

b. Using proposition we have A^n is a generator for the strong continuous semigroup $T_n(t) = T(t)|_{X_n}$ so according to Theorem 2.11 $A^n$is close and $D(A^n)$ is dense subset in X, hence $\overline{D(A^n)} = X$, each Cauchy sequence $\{f_k\}$ in $D(A^n)$ is a Cauchy sequence in X, so it is convergence in X, since X is a Banach space , Hence $\{f_k\}$ is convergence in $D(A^n)$ with the norm $\|.\|_{A^n} = \|.\|_n$ and $D(A^n)$ is a Banach space.

Exercise 4.3

$$\dot{u}(t) = Au(t), \quad u(0) = f,$$
$$T(h)f = e^{Ah}f = \sum_{n=0}^{\infty} \frac{(hA)^n}{n!} f = f + hAf + o(h^2),$$
$$F(h) = \frac{1}{h} R\left(\frac{1}{h}, A\right) = (I - Ah)^{-1},$$

Stability follows from (2.2). we have to deal with consistency. Let $f \in D(A)$ and note that

$$F(h) = AR\left(\frac{1}{h}, A\right) + I,$$

hold. Hence

$$F(h)f - T(h)f = R\left(\frac{1}{h}, A\right)Af + f - \left(f + hAf + \frac{h^2}{2}A^2 f + O(h^3)\right)$$
$$= \left(R\left(\frac{1}{h}, A\right)Af - hAf\right) - (\frac{h^2}{2}A^2 f + O(h^3)),$$
$$\Longrightarrow \quad \left\|\frac{F(h)f - T(h)f}{h}\right\| \le \left\|\frac{R\left(\frac{1}{h}, A\right)Af - hAf}{h}\right\| + o(h),$$

$$\Longrightarrow \quad \|F(h)f - T(h)f\| = o(h^2),$$

From Proposition 4.12 the assertion follows.

Exercise 4.4

Exercise 4.4.1

Let

$$y_{n+1} = y_n + hf(t_n, y_n), \qquad t_n = t_0 + nh,$$

we define

$$\phi(t, y, h) = f(t, y)$$

Then the numerical method can be rewritten as

$$y_{n+1} = y_n + h\phi(t_n, y_n, h),$$

since

$$\phi(t, y, 0) = f(t, y)$$

the explicit Euler method is consistent and we have

$$T_n(h) = \frac{y(t_{n+1}) - y(t_n)}{h} - f(t_n, y_n)$$

$$= y'(t_n) + \frac{h}{2}y''(t_n) + o(h^2) - f(t_n, y_n)$$

$$= \frac{h}{2}y''(\tau) \quad \tau \in (t_0, t_{max}).$$

Therefore, there exist a constant K such that

$$|T_n(h)| \leq Kh.$$

Then the explicit Euler method is first-order consistency.

Now, consider the explicit Runge's method

$$y_{n+1} = y_n + hf\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}f(t, y)\right),$$

Let us define

$$\phi(x, y, h) = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}f(t, y)\right).$$

Then the numerical method can be rewritten as

$$y_{n+1} = y_n + h\phi(t_n, y_n, h),$$

since

$$\phi(t, y, 0) = f(t, y),$$

the method is consistent and it is easily verified by Taylor series expansion that the truncation error is of the form

$$T_n(h) = \frac{1}{6}h^2\left[f_y(f_x + ff_y) + \frac{1}{4}(f_{xx} + 2ff_{xy} + f^2 f_{yy})\right] + o(h^3),$$

Therefore, there exist a constant M such that

$$|T_n(h)| \leq Mh^2.$$

Then the explicit Runge's method is second-order consistency.

Exercise 4.4.2

Consider the implicit Euler method

$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1}), \qquad t_n = t_0 + nh.$$

for the numerical solution of the initial value problem $y'(t) = y(t)$, $y(0) = 1$.

Let $t_0 = 0$, $t_{max} = 1$ and $n \in N, nh = 1$. After one step reads:

$$y_1 = y_0 + hy_1 \implies (1-h)y_1 = y_0 \implies y_1 = \frac{1}{1-h},$$

$$y_2 = y_1 + hy_2 \implies (1-h)y_2 = y_1 \implies y_2 = \left(\frac{1}{1-h}\right)^2,$$

Hence, the numerical solution after n steps

$$y_n = \left(\frac{1}{1-h}\right)^n,$$

then

$$\lim_{n\to\infty} y_n = \left(\frac{1}{1-\frac{1}{n}}\right)^n = e.$$

Exercise 4.4.3

$$y_{n+1} = y_n + h_n \sum_{i=1}^{s} b_i k_i,$$

$$b_i = \int_0^1 \ell_i(\tau)d\tau, \quad k_i = f(t_n + c_i h_n, y_n + h_n \sum_{j=1}^{s} a_{ij} k_j, \quad a_{ij} = \int_0^{c_i} \ell_j(\tau)d\tau.$$

a. Runge–Kutta method based on Gaussian rule

Let $c_1 = \frac{1}{2} + \frac{\sqrt{3}}{6}, c_2 = \frac{1}{2} - \frac{\sqrt{3}}{6}$, then

$$b_1 = \int_0^1 \ell_1(\tau)d\tau = \int_0^1 \frac{\tau - \left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right)}{\frac{\sqrt{3}}{3}} d\tau = \frac{3}{\sqrt{3}}\left[\frac{\tau^2}{2} - \left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right)\tau\right]_0^1 = \frac{1}{2},$$

$$b_2 = \int_0^1 \ell_2(\tau)d\tau = \int_0^1 \frac{\tau - \left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right)}{-\frac{\sqrt{3}}{3}} d\tau = -\frac{3}{\sqrt{3}}\left[\frac{\tau^2}{2} - \left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right)\tau\right]_0^1 = \frac{1}{2},$$

$$a_{11} = \int_0^{c_1} \ell_1(\tau)d\tau = \frac{3}{\sqrt{3}}[\frac{\tau^2}{2} - (\frac{1}{2} - \frac{\sqrt{3}}{6})\tau]_0^{c_1} = \frac{1}{4},$$

$$a_{22} = \int_0^{c_2} \ell_2(\tau)d\tau = -\frac{3}{\sqrt{3}}[\frac{\tau^2}{2} - (\frac{1}{2} + \frac{\sqrt{3}}{6})\tau]_0^{c_2} = \frac{1}{4},$$

$$a_{12} = \int_0^{c_1} \ell_2(\tau)d\tau = \frac{-3}{\sqrt{3}}[\frac{\tau^2}{2} - (\frac{1}{2} + \frac{\sqrt{3}}{6})\tau]_0^{c_1} = \frac{\sqrt{3}}{6} + \frac{1}{4},$$

$$a_{21} = \int_0^{c_2} \ell_1(\tau)d\tau = \frac{3}{\sqrt{3}}\left[\frac{\tau^2}{2} - \left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right)\tau\right]_0^{c_2} = \frac{1}{4} - \frac{\sqrt{3}}{6}.$$

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$\frac{1}{2} + \frac{\sqrt{3}}{6}$	$\frac{1}{4}$	$\frac{1}{4} + \frac{\sqrt{3}}{6}$
$\frac{1}{2} - \frac{\sqrt{3}}{6}$	$\frac{1}{4} - \frac{\sqrt{3}}{6}$	$\frac{1}{4}$
	$\frac{1}{2}$	$\frac{1}{2}$

b. Runge–Kutta method based on Radau II A rule

Let $c_1 = \frac{1}{3}, c_2 = 1$, then

$$b_1 = \int_0^1 \ell_1(\tau)d\tau = \int_0^1 \frac{\tau - 1}{-\frac{2}{3}} d\tau = \frac{3}{2}\left[\tau - \frac{\tau^2}{2}\right]_0^1 = \frac{3}{4},$$

$$b_2 = \int_0^1 \ell_2(\tau)d\tau = \int_0^1 \frac{\tau - \left(\frac{1}{3}\right)}{\frac{2}{3}} d\tau = \frac{3}{2}\left[\frac{\tau^2}{2} - \left(\frac{1}{3}\right)\tau\right]_0^1 = \frac{1}{4},$$

$$a_{11} = \int_0^{\frac{1}{3}} \ell_1(\tau)d\tau = \frac{3}{2}\left[\tau - \frac{\tau^2}{2}\right]_0^{\frac{1}{3}} = \frac{5}{12},$$

$$a_{22} = \int_0^{c_2} \ell_2(\tau)d\tau = \frac{3}{2}\left[\frac{\tau^2}{2} - \left(\frac{1}{3}\right)\tau\right]_0^1 = \frac{1}{4},$$

$$a_{12} = \int_0^{\frac{1}{3}} \ell_2(\tau)d\tau = \frac{3}{2}\left[\tau - \frac{\tau^2}{2}\right]_0^{\frac{1}{3}} = -\frac{1}{12},$$

$$a_{21} = \int_0^{c_2} \ell_1(\tau)d\tau = \frac{3}{2}\left[\tau - \frac{\tau^2}{2}\right]_0^1 = \frac{3}{4}.$$

$$
\begin{array}{c|cc}
\frac{1}{3} & \frac{5}{12} & -\frac{1}{12} \\
1 & \frac{3}{4} & \frac{1}{4} \\
\hline
 & \frac{3}{4} & \frac{1}{4}
\end{array}
$$

Exercise 4.4.5

$$y_{n+1} = y_n + \frac{h_n}{2}\big(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\big),$$

$$y_{n+1} = y_n + h_n \sum_{i=1}^{2} b_i k_i \qquad \Longrightarrow\ b_1 = b_2 = \frac{1}{2},$$

$$k_1 = f(t_n, y_n) \quad \Longrightarrow \quad c_1 = 0, \quad a_{11} = a_{12} = 0,$$

$$k_2 = f(t_{n+1}, y_{n+1}) = f\left(t_n + h, y_n + \frac{h}{2}k_1 + \frac{h}{2}k_2\right) \quad \Longrightarrow \quad c_1 = 1, \quad a_{21} = a_{22} = \frac{1}{2}.$$